ACOUSTIC RADIATION FROM FLUID LOADED RECTANGULAR PLATES

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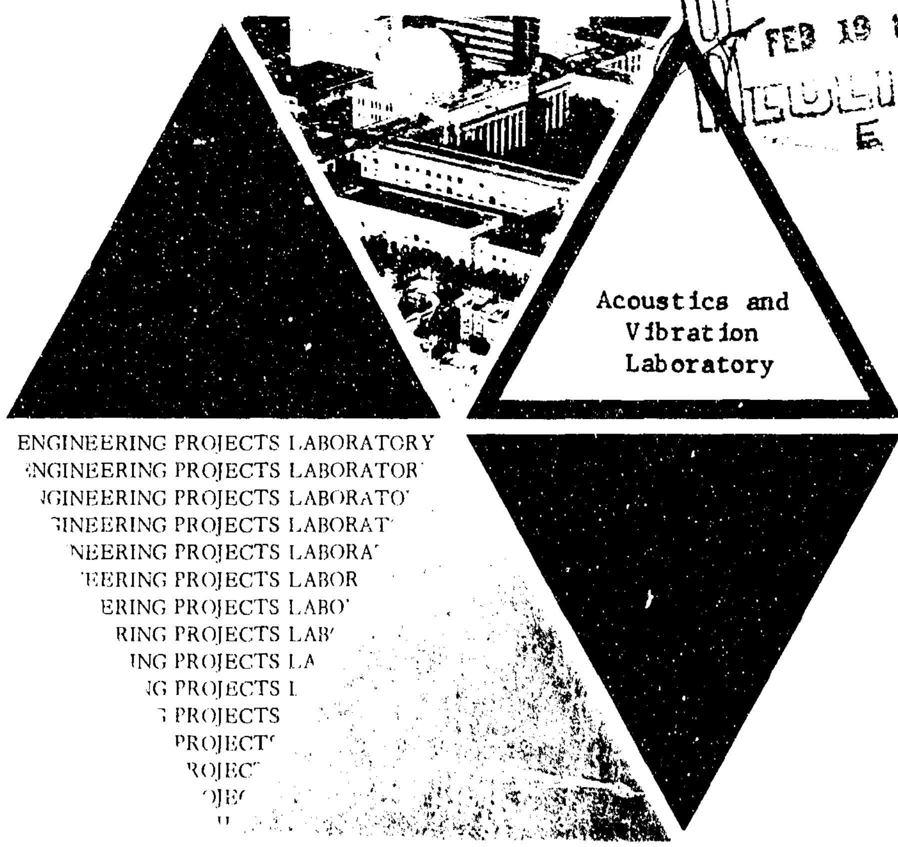

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Abstract

The acoustic radiation into a fluid filled infinite half-space from a randomly excited, thin rectangular plate inserted in an infinite baffle is discussed. The analysis is based on the in vacuo modes of the plate. The modal coupling coefficients are evaluated approximately at both low and high (but below acoustic critical) frequencies. An approximate solution of the resulting infinite set of linear simultaneous equations for the plate modal velocity amplitudes is obtained in terms of modal admittances of the plate-fluid system. These admittances describe the important modal coupling due to both fluid inertia and radiation damping effects. The effective amount of coupling, and hence the effective radiation damping acting on a mode, depends on the relative magnitudes of the structural damping, i.e., on the widths of the modal resonance peaks, and the frequency spacing of the resonances. Expressions are obtained for the spectral density of the radiated acoustic power for the particular case of excitation by a turbulent boundary layer.

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1. Introduction

Much research has recently been done on the response to random excitation of structures vibrating in air. In general, and certainly as far as most practical situations are concerned, the response in this case is effectively the in vacuo response, no consideration of the interaction of the structural vibrations with the associated sound field being necessary. The acoustic radiation, if required, can then be estimated from the already determined structural response.

Light fluid loading effects have also been included in some analyses. Much of this work is based on the statistical energy method of Lyon and Maidanik[1]. Maidanik[2] has used the method to estimate the radiation from finite panels vibrating in air. More recently, Leehey[3] and Davies[4] have discussed its application to turbulent boundary layer excited panels. The present analysis is to a certain extent an extension of some of this work.

Previous work concerning water loading effects has concentrated on spherical shells, infinite cylindrical shells and infinite thin plates (see, for example, Junger[5], which contains a large number of additional references, and Maidanik[6]). In each of these cases the interaction problem is simplified because the in vacuo normal modes of the structure are maintained when the structure is submerged in water, as the acoustic field can also be expanded in the same series of characteristic functions or modes. For the infinite plate, the series is, of course, replaced by an integral over a continuous spectrum of wavenumbers.

In situations, such as that to be considered here, where the structural response is characterised by a discrete wavenumber spectrum and the acoustic field by a continuous wavenumber spectrum, the in vacuo normal modes are not retained. However, the expansion of the velocity response of a structure in terms of its in vacuo modes is still valid. It is convenient still to refer to these functions as modes and to talk of the resonance frequencies of these modes, in which case although we do not refer to a frequency associated with some natural mode of vibration, we still imply a frequency associated with the maximum value of the amplitude response of a characteristic function. An essential feature of this problem now becomes the coupling together of the in vacuo modes by the structure fluid interaction. Although we discuss here the particular case of radiation into a fluid filled semi-infinite space from a rectangular plate in an infinite rigid baffle, the arguments given concerning the amount and the effect of the modal coupling induced can obviously be applied to other geometries. What we attempt in this paper is hardly a complete solution of the coupled problem; the modal interactions are far too complicated; but rather, after a considerable but necessary series of approximations, we determine and interpret the most important features of the structure-fluid interaction and their effect on the response of the system.

We consider a simply supported thin rectangular plate inserted in an infinite rigid baffle and fluid loaded on one side. The normal

vibration velocity field of the plate is expanded in a series of the in vacuo normal modes or characteristic functions of the plate. This approach leads, because of the structure-fluid interaction, to an infinite set of simultaneous linear algebraic equations to be solved for the infinite number of unknown modal response amplitudes. These equations are obtained in section 2, below. Furthermore, many of the coefficients in these equations; those associated with the fluid loading terms; are defined by integrals which themselves can only be evaluated approximately for various regimes of frequency. Some of these coefficients have been evaluated by Maidanik(2) and also by Davies(4). We will continue to refer to those discussed by Maidanik(2) as modal radiation coefficients as they are a measure of how efficiently a particular modal shape radiates when no other modes are excited. However, we also require the modal coupling coefficients connecting the vibration of one plate mode with that of other plate modes because of the plate-fluid interaction. These additional coefficients are obtained, at least asymptotically at low and high frequencies, and discussed in section 3. The previously determined modal radiation coefficients can obviously be obtained as special cases of the modal coupling coefficients. The real parts of the coefficients are associated with a radiation damping effect on the plate response: the imaginary parts lead to a virtual mass to be added to the mass of the plate, hence causing a decrease in the modal resonance frequencies.

In sections 4 and 5 the solution of the resulting infinite set of modal equations is discussed: for low frequencies in section 4, and for high frequencies (but below the acoutic critical frequency) in section 5. At low frequencies, $k_o\ell_1$, $k_o\ell_3$ $<$ $\pi/2$ (where k_o is the acoustic wavenumber and ℓ_1,ℓ_3 are the dimensions of the plate), all modes have similar radiation characteristics and the modal equations are all of the same form. An approximate solution of the set of equations is discussed and modal admittances for the plate-fluid system obtained which contain the important coupling effects. The radiated power spectral density is then discussed. A simple expression is obtained in terms of the modal radiation coefficients, the modal components of the correlation function of the applied force and the modal admittance functions. It is assumed that the applied force is such that it causes no additional modal coupling. This requires that the typical correlation lengths of the forcing field be much less than the panel dimensions, a condition that is satisfied in many practical applications. The case when this condition does not hold is discussed briefly in an appendix. The modal admittances contain the virtual mass terms and additional damping terms due to the fluid loading. The inertia coupling terms are small. Because of the coupling, the radiation damping term is found to be itself a summation over many modes. Now, if the total damping is assumed small, the power radiated in a narrow band of frequencies is mainly from the modes resonantly excited at frequencies within the

band. Furthermore, we need only consider the coupling between resonant modes in the band. It is shown that the magnitude of the radiation damping is determined by the amount of modal coupling, that is, by the numbers of modes that interact. This in turn depends on the relative magnitudes of the structural damping, a measure of the width of the resonance peaks, and the frequency spacing between resonance peaks. Under the assumption of light damping there can be no likelihood of power flow between modes, that is, no coupling, if the resonance peaks do not overlap. This dependence of the radiation damping on the structural damping is discussed in section 4. Estimates are obtained of the radiation damping under light (hence no coupling) and heavy structural damping. These are then used in the expressions for the spectral density of radiated power obtained by averaging over the resonant modes in narrow frequency bands.

At high frequencies, $k_o\ell_1, k_o\ell_3 \gg \pi$, the modal interactions are more complicated, and an analysis of the modal coupling effects correspondingly more difficult. Maidanik[2] has shown that at these frequencies the modes can be divided into three groups according to their radiation characteristics, namely, edge, corner and acoustically fast modes (see section 3, below). We estimate, in narrow bands of frequency, the total coupling between different types of modes. Since we assume small damping, it is sufficient when considering edge and corner modes in the estimates to include only

resonant modes. We restrict the analysis to frequencies below the acoustic critical frequency, that is, the resonant modes we consider all have wavespeeds on the plate less than the acoustic wavespeed, and thus do not consider the case of resonantly excited acoustically fast modes. This is hardly restrictive in practice; the acoustic critical frequency for a 1/4" steel plate in water is about 400,000 Hertz. However, as the radiation efficiency of the acoustically fast modes is high, it is necessary to determine whether or not the contribution from these modes to the radiated power at any frequency is of importance even when the modes are non-resonantly excited. These modes are thus included in the analysis. Our estimates of the modal coupling effects in narrow frequency bands thus include the interactions between resonantly excited edge and corner modes and non-resonantly excited acoustically fast modes. We find, however, that in most cases the acoustically fast modes are not sufficiently highly excited either by their being coupled to resonant edge modes or by the acoustically fast mode component of the external forcing field to give any considerable contribution to the radiated field. Expressions for the spectral density of radiated power are again obtained. The radiation damping of a mode is again found in the form of a summation over many modes because of the coupling effect. This summation is evaluated as in the low frequency case.

In section 6 the directivity of the radiated field is briefly discussed.

In section 7 numerical estimates of the radiated power spectral density are made assuming the plate is excited by a turbulent boundary layer. Corcos[7] model of the correlation function of the wall pressure field is used. The estimates are compared with the spectral density obtained neglecting fluid loading. A wide range of values of the structural damping is used to demonstrate the dependence of the modal radiation damping on the structural damping.

Finally in an appendix, we discuss the case when the modes are also coupled together by the external field. A simple expression for the radiated power spectral density can still be obtained at low frequencies in terms of the modal coupling coefficients, the modal admittance functions already discussed, and Powell's[8] modal joint acceptances. It would seem that the two forms of coupling act independently, and can be dealt with separately.

2. The Coupled Equations of Motion

A simply supported thin rectangular panel of length ℓ_1 and width ℓ_3 is inserted in a flat infinite rigid baffle. The acoustic field radiated into the dense fluid* in the semi-infinite space on one side of the panel and baffle will be considered. It is assumed that neither the panel vibration nor the acoustic field affect the applied external force.

Rectangular coordinates $(x_1, x_2, x_3) \equiv (\underline{x}, x_2)$ are chosen as usual, x_2 being normal to the panel and the origin being at one corner of the

*e.g. water

panel. The fluid fills the region $x_2 > 0$ (see Figure 1). The equation for thin panel normal displacement $y(\underline{x},t)$ driven by a pressure field $p(\underline{x},t)$ is:

$$D\nabla^4 y + m_p \beta \frac{\partial y}{\partial t} + m_p \frac{\partial^2 y}{\partial t^2} = p(\underline{x},t) - P(\underline{x}, x_2 = 0, t) , \qquad 2.1$$

where $P(\underline{x}, x_2, t)$ is the acoustic pressure generated by the motion of the panel, and D and m_p are the flexural rigidity and mass per unit area of the panel, respectively. β is a coefficient introduced to account for mechanical damping of the panel.

Following reference 4, we consider the frequency Fourier transform of equation 2.1 and expand the panel normal velocity displacement $v(\underline{x},\omega)$ in terms of the normalised characteristic functions

$$\psi_{mn}(\underline{x}) = \frac{2}{\sqrt{A_p}} \sin k_m x_1 \sin k_n x_3 ,$$

where $k_m = \frac{m\pi}{\ell_1}$, $k_n = \frac{n\pi}{\ell_3}$ and $A_p = \ell_1 \ell_3$. The modal equation for the frequency Fourier transform of panel velocity is thus obtained:

$$\left[D k_{mn}^4 - \omega^2 m_p - i\omega^2 m_p \eta_{mn} \operatorname{sgn} \omega \right] v_{mn}(\omega) = -i\omega p_{mn} + i\omega P_{mn} . \qquad 2.2$$

where $k_{mn}^2 = k_m^2 + k_n^2$, η_{mn} is a modal structural loss factor and p_{mn} and P_{mn} are of similar form and defined by

$$p_{mn} = \int_{A_p} p(\underline{x}, \omega)\, \psi_{mn}(\underline{x})\, d\underline{x}$$

$$P_{mn} = \int_{A_p} P(\underline{x}, 0, \omega)\, \psi_{mn}(\underline{x})\, d\underline{x} \quad .$$

P_{mn} can be expressed in terms of the modal velocity amplitudes v_{mn} using the wave equation for $P(\underline{x}, x_2, \omega)$ in the acoustic medium, and the boundary condition in the plane of the plate, namely

$$\left.\frac{\partial P}{\partial x_2}\right|_{x_2 = 0} = i \omega \rho_0 v(\underline{x}, \omega) \quad .$$

After some rearrangement, we can obtain $P_{mn}(\omega)$ in the form

$$P_{mn}(\omega) = \sum_{q,r=1}^{\infty} R_{mnqr}(\omega)\, v_{qr}(\omega) \quad ,$$

where R_{mnqr} is a coupling coefficient connecting the (m,n) mode and the (q,r) mode defined by

$$R_{mnqr}(\omega) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} Z(\underline{k}, \omega)\, S_{mn}(\underline{k})\, S_{qr}^{*}(\underline{k})\, d\underline{k} \quad . \qquad 2.3$$

Here, $Z(\underline{k},\omega)$ is the radiation impedance

$$Z(\underline{k},\omega) = \rho_o c_o \left(1 - \frac{k^2}{k_o^2}\right)^{-\frac{1}{2}}$$

with $k = |\underline{k}|$, $k_o = \omega/c_o$ and $S_{mn}(\underline{k})$ is a shape function defined by

$$S_{mn}(\underline{k}) = \int_{A_p} \psi_{mn}(\underline{x})\, e^{i\underline{k}.\underline{x}}\, d\underline{x} \quad . \qquad 2.4$$

Pr...ious results using light fluid loading approximations have only included the term R_{mnmn} and have neglected the modal coupling terms. It is shown below that many of the modal coupling coefficients are equal in magnitude to the modal radiation coefficients.

The equations of motion can now be written in the form

$$\left[D k_{mn}^4 - \omega^2 m_p - i\omega^2 m_p \eta_{mn} \operatorname{sgn}\omega\right] v_{mn}(\omega) - i\omega \sum_{q,r} R_{mnqr} v_{qr} = -\omega p_{mn} \quad . \qquad 2.5$$

In the following sections approximate values of the radiation coefficients R_{mnqr} are obtained for both high and low frequencies, and the solution of the system of coupled equations 2.5 is discussed.

3. The Radiation Coefficients

Values of the modal radiation coefficients have been obtained in references 2 and 4 for various frequency regimes. Maidanik's[2] classification of the modal radiation coefficients is shown in Figure 2. The modal wavenumbers are at fixed lattice points in wavenumber space. At any frequency ω we define the acoustic wavenumber $k_o = \omega/c_o$, which, in $\underline{k}$-space, divides the modes into those with $k_{mn} < k_o$, thus having a wavespeed on the plate greater than the acoustic wavespeed and hence called acoustically fast modes, and those with $k_{mn} > k_o$ called acoustically slow modes. The acoustically slow modes can be further divided into edge and corner modes. These are further discussed at the end of this section.

The types of approximations used in references 2 and 4 are also suitable for evaluating many of the cross-coupling coefficients. However, for some high frequency results Lighthill's[9] methods for the asymptotic evaluation of Fourier transforms of generalised functions are used.

The shape functions used in the integral for R_{mnqr} are:

$$S_{mn}(\underline{k}) = \int_{A_p} \psi_{mn}(\underline{x})\, e^{i\underline{k}.\underline{x}}\, d\underline{x}$$

$$= \frac{2k_m k_n}{A_p(k_1^2 - k_m^2)(k_3^2 - k_n^2)} \left[(-1)^m e^{-ik_1\ell_1} - 1\right]\left[(-1)^n e^{-ik_3\ell_3} - 1\right]$$

The integrand in equation 2.3 thus includes terms of the form

$$1 + (-1)^{q+m} - (-1)^q e^{ik_1\ell_1} - (-1)^m e^{-ik_1\ell_1}$$

which has the effective value:

$$\begin{cases} 2(1 - (-1)^m \cos k_1\ell_1) \quad , & \text{if m and q are either both odd or both even} \\ 0 \quad , & \text{otherwise} \end{cases}$$

where we have noted that the rest of the integrand is even in k_1; the $\sin k_1\ell_1$ term in the integrand thus vanishing. It follows that R_{mnqr} is only non-zero when both m and q and n and r have the same parity, where by parity we mean here that m and q(and n and r) must be either both even or both odd. Each mode is thus coupled to at most only one quarter of all the other modes.

The radiation coefficients will be written:

$$R_{mnqr} = S_{mnqr} + i\, T_{mnqr}$$

$$= \frac{64\, \rho_o c_o}{(2\pi)^2 A_p} k_m k_n k_q k_r k_o \iint_0^\infty \frac{\left(1-(-1)^m \cos k_1 \ell_1\right)\left(1-(-1)^n \cos k_3 \ell_3\right) \quad dk_1\, dk_3}{(k_1^2-k_m^2)(k_1^2-k_q^2)(k_3^2-k_n^2)(k_3^2-k_r^2)(k_o^2-k_1^2-k_3)^{1/2}} \tag{31}$$

The real part of the integral, S_{mnqr}, involves integrations over values $|\underline{k}| < k_o$. We note that the only singularity of the integrand is the square root singularity. Typical values of the function

$$I_{mq}(k_1) = \frac{\left(1-(-1)^m \cos k_1 \ell_1\right)}{(k_1^2-k_m^2)(k_1^2-k_q^2)}$$

are shown in Figure 3. The function is such that

$$\int_0^\infty I_{mq}(k_1)\, dk_1 = 0$$

$$\text{if} \quad m \neq q .$$

We consider the term S_{mnqr} first. Various approximations are necessary for different types of modes. Evaluation of the various coefficients is conveniently divided into three parts.

a) $k_n, k_r >> k_o$ (with similar results applying to modes with $k_m, k_q >> k_o$.) $k_o\ell_1, k_o\ell_3 >> \pi$.

We make the approximation $k_3^2 - k_n^2 \sim -k_n^2$. The k_3 integration can then easily be performed leaving the approximate result:

$$S_{mnqr} = \frac{8\rho_o c_o}{\pi A_p} \frac{k_m k_q}{k_n k_r} k_o \int_0^{k_o} I_{mq}(k_1)\, dk_1 \quad .$$

If $k_m, k_q < k_o$ we can write the integral as $\int_0^\infty - \int_{k_o}^\infty$ to obtain the result:

$$S_{mnqr} = \frac{8\rho_o c_o}{\pi A_p} \frac{k_m k_q}{k_n k_r} k_o \left(\frac{\pi \ell_1 \delta_{mq}}{4 k_m^2} - \frac{1}{3 k_o^2} \right) \qquad 3.2$$

where δ_{mq} is a Kronecker delta. The last term here will be neglected as being of higher order in powers of $(k_o\ell)^{-1}$.

If $k_m < k_o$, $k_q > k_o$, the integral is written

$$\int_0^{k_o} I_{mq}(k_1)\, dk_1 = -\frac{1}{k_q^2} \int_0^{k_o} \frac{\left(1 - (-1)^m \cos k_1 \ell_1\right)}{k_1^2 - k_m^2}\, dk_1 = \frac{1}{k_q^2 k_o}$$

giving

$$S_{mnqr} = \frac{8\rho_o c_o}{\pi A_p} \frac{k_m}{k_n k_r k_q} .$$

The case k_m, $k_q > k_c$ is treated more accurately in part (c) without the restriction $k_o \ell_1$, $k_o \ell_3 >> \pi$

b) $k_m = k_q < k_o$, $k_o \ell_1$, $k_o \ell_3 >> \pi$

Following Maidanik[2], and earlier Kraichnan[10] (his equation 5.5) a delta function approximation can be used in the form

$$I_{mm}(k_1) \approx \frac{\pi \ell_1}{4 k_m^2} \delta(k_1 - k_m)$$

leading to the expression

$$S_{mnmr} \approx \frac{4\rho_o c_o}{\pi A_p} k_o \ell_1 k_n k_r \int_0^{k_o} I_{nr}(k_3) \frac{dk_3}{(k_o^2 - k_3^2)^{1/2}} \qquad 34$$

For $k_n = k_r$, a similar delta function approximation leads directly to the acoustically fast mode radiation coefficient $S_{mnmn} = \rho_o c_o$.

For k_n , $k_r > k_o$ we obtain the edge mode result of equation 3.2.

For the coupling coefficient between two acoustically fast modes or an edge mode and an acoutically fast mode we require an estimate of the integral

$$I = \int_0^{k_0} \frac{\left(1 - (-1)^m \cos k_3 \ell_3\right) \, dk_3}{\left(k_3^2 - k_r^2\right) \left(k_0^2 - k_3^2\right)^{1/2}}$$

for $k_r < k_o$. As this integral is relevant only for edge or acoustically fast modes, it is sufficient to evaluate it for large values of $k_o \ell_3$. We can treat the terms 1 and $(-1)^n \cos k_3 \ell_3$ separately if we treat the separate integrals as Cauchy principal values. The first integral is zero in this case and we are left with an integral which we can write, following Lighthill[9], as

$$I = (-1)^m \int_{-\infty}^{\infty} f(z) \, e^{i(k_0 \ell_3) z} \, dz$$

where

$$f(z) = \frac{1}{k_o^2}\,\frac{H(z+1)\,H(1-z)}{\left(z^2 - \frac{k_r^2}{k_o^2}\right)\left(1 - z^2\right)^{1/2}} \, ,$$

and H is the unit step function. The function f has singularities at $z = \pm k_r/k_o$ and $z = \pm 1$. The main contributions to the integral at large values of $k_o\ell_3$ can be estimated from the Laurent series expansion of f(z) about its singularities. We find, as expected, no contribution from the singularities at $z = \pm k_r/k_o$, and from the square root singularities obtain the asymptotic result

$$I \simeq -(-1)^r \frac{1}{k_o^2 - k_r^2}\left(\frac{\pi}{2k_o\ell_3}\right)^{1/2} \cos\left(k_o\ell_3 - \frac{\pi}{4}\right) + o\left(|k_o\ell_3|^{-3/2}\right) .$$

Equation 3.4 can now be evaluated for all cases. The results are summarized below.

c) A result valid for all corner modes at all frequencies is obtained from equation 3.1 by writing it in the form

$$S_{mnqr} = \frac{8\rho_0 c_0}{\pi^2 A_p} \frac{k_0}{k_m k_q k_n k_r} \iint_0^{|\underline{k}|<k_0} \frac{\left(1-(-1)^m \cos k_1 \ell_1\right)\left(1-(-1)^n \cos k_3 \ell_3\right)}{\left(k_0^2 - k_1^2 - k_3^2\right)^{1/2}}\, d\underline{k}$$

$$= \frac{8\rho_0 c_0}{\pi A_p} \frac{k_0^2}{k_m k_q k_n k_r} \left\{ 1 - (-1)^m \frac{\sin k_0 \ell_1}{k_0 \ell_1} - (-1)^n \frac{\sin k_0 \ell_3}{k_0 \ell_3} + (-1)^{m+n} \frac{\sin k_0 \sqrt{\ell_1^2 + \ell_3^2}}{k_0 \sqrt{\ell_1^2+\ell_3^2}} \right\} . \qquad 3.6$$

We note that because of the parity of (m,n) and (q,r) the $(-1)^m$ and $(-1)^n$ terms are interchangeable with $(-1)^q$ and $(-1)^r$ terms, respectively. S_{mnqr} is thus symmetric in m and q and n and r.

Values of the real parts of the modal coupling coefficients can be summarized as follows:

If (m,n) is an X-type edge mode and...

i) (q,r) is an edge mode ,

$$S_{mnqr} = \frac{2\rho_0 c_0}{A_p} \frac{k_0 \ell_1}{k_n k_r} \varepsilon_{mq} ,$$

ii) (q,r) is a corner mode,

$$S_{mnqr} = \frac{2\rho_0 c_0}{\pi A_p} \frac{k_m}{k_n k_r k_q} , \qquad 3.8$$

iii) (q,r) is an acoustically fast mode,

$$S_{mnqr} = (-1)^n \frac{4\rho_0 c_0}{\pi A_p} \frac{k_r}{k_n k_0^2} k_0 \ell_1 \left(\frac{\pi}{2 k_0 \ell_3}\right)^{1/2} \cos\left(k_0 \ell_3 - \frac{\pi}{4}\right) \delta_{mq} . \qquad 3.9$$

If both (m n) and (q,r) are acoustically fast modes

$$S_{mnqr} = \rho_0 c_0 \delta_{mq} \delta_{nr} + (-1)^n \frac{4\rho_0 c_0}{\pi A_p} \frac{k_n k_r}{k_0^4} k_0 \ell_1 \left(\frac{\pi}{2 k_0 \ell_3}\right)^{1/2} \cos\left(k_0 \ell_3 - \frac{\pi}{4}\right) \delta_{mq}$$

$$+ (-1)^m \frac{4\rho_0 c_0}{\pi A_p} \frac{k_m k_q}{k_0^4} k_0 \ell_3 \left(\frac{\pi}{2 k_0 \ell_1}\right)^{1/2} \cos\left(k_0 \ell_1 - \frac{\pi}{4}\right) \delta_{nr} .$$

3.10

Equation (3.6) gives values of S_{mnqr} when both modes are corner modes which are valid at all frequencies.

We note that the edge, corner and acoustically fast modal radiation coefficients can be obtained as special cases of the above results.

Only the dominant terms in the expressions for different modes have been retained. At this point we make no comparisons of the relative magnitude of the various coupling terms, leaving this discussion until sections 4 and 5. The relative importance of the coupling terms will obviously be influenced by the numbers of each of the various types of modes which are coupled.

In a similar way we estimate the mass loading coupling term T_{mnqr}. The integration involved is over all wavenumbers greater than the acoustic wavenumber. The required value of the square root in equation (3.1) is now $-i(k_1^2 + k_3^2 - k_0^2)^{1/2}$. Thus, for all acoustically slow modes the range of integration includes all the modal wavenumbers. It is obvious in this case from the nature of the integrand that the largest coefficients are those having either, or both, m=q and n=r. We consider first the coupling coefficient of two X-type modes. The range of integration can be divided into the three regions

$$\iint\limits_{|\underline{k}| \geq k_0} d\underline{k} = \int_{k_0}^{\infty} dk_3 \int_{0}^{\infty} dk_1 + \int_{0}^{k_0} dk_3 \int_{k_0}^{\infty} dk_1 + \int_{0}^{k_0} dk_3 \int_{\sqrt{k_0^2 - k_3^2}}^{k_0} dk_1 .$$

For two X-type edge modes, it is easily seen that the first of these terms is the dominant one. We obtain in this case

$$T_{mmr} = -\frac{4\rho_0 c_0}{\pi A_p} k_n k_r k_0 \ell_1 \int_{k_0}^{\infty} \frac{I_{nr}(k_3)}{\left(k_3^2 - (k_0^2 - k_m^2)\right)^{1/2}} \, dk_3 \, . \qquad 3.11$$

We, thus, obtain integrals of a form similar to that in equation (3.5). We again treat the terms 1 and $(-1)^n \cos k_3 \ell_3$ separately as Cauchy principal values. The square root singularity leads to the same asymptotic form as before. However, we now obtain a finite contribution from the term

$$-\int_{k_0}^{\infty} \frac{1}{(k_3^2 - k_n^2)\left(k_3^2 - (k_0^2 - k_m^2)\right)^{1/2}} \, dk_3 \, .$$

This gives the dominant term. It is sufficiently accurately estimated as

$$-\frac{1}{k_n^2} \log \frac{2k_n}{k_0} \, .$$

The terms arising from the cos $k_3\ell_3$ integration are now not required. The modal mass coupling term for two X-type edge modes is obtained in the form:

$$T_{mnmr} = -\frac{4\rho_0 c_0}{\pi A_p} \frac{k_0 \ell_1}{k_n k_r} \log \frac{2k_{mn}}{k_0}, \qquad 3.12$$

where we have made the further approximation

$$\log \frac{2k_n}{k_0} = \log \frac{2k_r}{k_0} \qquad 3.13$$

We will be concerned in the fol owing sections with the coupling between resonant edge modes only, that is with modes close together in wavenumber space. For these modes, expression (3.13) is approximately true.

In a similar way we may obtain the mass coupling coefficients between other types of modes. We note that when both modes are corner modes we obtain in place of the integral in (3.1 , proximate form

$$\int_0^\infty \frac{I_{nr}(k_3)\, dk_3}{(k_3^2 + k_m^2)^{1/2}} \approx \frac{1}{k_{mn}^2 \; k_{mr}^2} \qquad (n \neq r) \qquad 3.14$$

iii) (q,r) is an acoutically fast mode

$$T_{mnqr} = -\frac{4\rho_o c_o}{A_p} k_o \ell_1 \frac{k_r}{k_n k_o^2} \delta_{mq} \qquad 3.18$$

When both (m,n) and (q.r) are corner modes we obtain:

$$T_{mnqr} = -\frac{4\rho_o c_o}{\pi A_p} k_o \ell_1 \frac{k_n k_r}{k_{mn}^2 k_{mr}^2} \delta_{mq} - \frac{4\rho_o c_o}{\pi A_p} k_o \ell_3 \frac{k_m k_q}{k_{mn}^2 k_{qn}^2} \delta_{nr} \qquad 3.19$$

For all modes we also have the result:

$$T_{mnmn} = \begin{cases} -\rho_o c_o \dfrac{k_o}{k_{mn}} , & k_{mn} > k_o \\ 0 , & k_{mn} < k_o \end{cases} \qquad 3.20$$

The mass coupling between two acoustically fast modes is negligibly small.

Again as in the expression for S_{mnqr} only the dominant terms in the expressions for T_{mnqr} have been retained.

Expressions (3.6) to (3.10) for S_{mnqr} and expressions (3.16) to (3.20) for T_{mnqr} show a marked contrast in the nature of the coupling induced by the radiation terms and the mass loading terms, respectively. A physical basis for the nature of the radiation coupling coefficients

We also note that in both expressions (3.11) and (3.14) when n=r the delta function approximation leads directly to the result

$$T_{mnmn} = - \rho_o c_o \frac{k_o}{k_{mn}} \qquad 3.15$$

This result is true for acoustically slow modes. Further inspection shows that T_{mnmn} is negligibly small for acoustically fast modes.

Values of the imaginary parts of the modal coupling coefficients can be summarized as follows (compare equations (3.7) forward):

If (m,n) is an X-type edge mode and...

i) (q,r) is an edge mode

$$T_{mnqr} = - \frac{4\rho_o c_o}{\pi A_p} \frac{k_o \ell_1}{k_n k_r} \log \frac{2k_{mn}}{k_o} \delta_{mq} - \frac{4\rho_o c_o}{\pi A_p} k_o \ell_3 \frac{k_m k_q}{k_{mn}^2 k_{qn}^2} \delta_{nr} \qquad 3.16$$

ii) (q,r) is a corner mode

$$T_{mnqr} = - \frac{4\rho_o c_o}{\pi A_p} k_o \ell_3 \frac{k_m k_q}{k_{mn}^2 k_{qn}^2} \delta_{nr} \qquad 3.17$$

can be given in terms of the volume velocity cancellation effects described by Maidanik[2]. We refer to Figure 4. When the acoustic wavelength λ_o is much greater than the modal wavelength λ_n the radiation character of the mode can be represented by a series of pistons of width $\lambda_n/2$, there being a phase change of 180^o between neighboring pistons. Again as $\lambda_o > \lambda_n$, the combined radiation from two adjacent half pistons is of dipole character. The modal radiation character is thus described by a series of dipoles, leaving at each end monopoles of width $\lambda_n/4$. The dominant radiation is from these edge monopoles. These monopoles are themselves coupled if $\lambda_o > \ell_3$. However, the dipoles further combine to higher order multipoles in this case and the dominant radiation is again from the edges. If, on the other hand, $\lambda_o < \lambda_n$, there is no such cancellation, as the radiation from the mode shape is essentially uncoupled along the entire length of the plate. Thus, for example, an X-type edge mode is so called ($\lambda_m > \lambda_o$, $\lambda_n < \lambda_o < \ell_3$) because the volume velocity cancellation in the x_3 direction and the lack of cancellation in the x_1 direction lead to dominant radiation from strips of width $\lambda_n/4$ along the edges $x_1=0$ and $x_1=\ell_1$.

We may treat the modal radiation coupling coefficient S_{mnqr} as a measure of the contribution from the (q,r) modal plate velocity to the (m,n) modal component of the acoustic field. We first consider equation (3.7), where both modes are X-type edge modes. The contribution

is large when the mode numbers in the x_1 direction are the same. Along the length of the strip radiators we require, for large coupling, that there be no inter-modal velocity cancellations, that is, the strips must vibrate in the same mode shape along their length. Changes in the width of the strip are of negligible importance. Similar arguments also explain the presence of the Kronecker delta functions in equations (3.9) and (3.10). However, when we consider the radiation coupling of a corner mode with either another corner mode or an edge mode (equations (3.6) and (3.8), respectively) we find, of course, that the only contribution from the (q,r) modal plate velocity to the (m,n) component of the acoustic field now comes from the uncancelled monopole radiators in each corner. The only criterion for there to be a coherent contribution is that these corner radiators be in phase. This is controlled solely by the parity of (m,n) and (q,r).

The imaginary parts of the modal coupling coefficients, T_{mnqr}, lead to virtual mass terms to be added to the mass of the plate. Volume velocity cancellation effects are now of no consequence; the inertia terms act over the whole area of the plate. This is demonstrated by the fact that T_{mnmn} is the same for all acoustically slow modes and that even the mass coupling terms T_{mnqr} in equations (3.16) to (3.19) are essentially of the same form, irrespective of the division by radiation characteristics into edge and corner modes. For there to be large inertia coupling between modes we require that two mode numbers be the same, so that in one direction the modes vibrate in the same shape. This is symbolised by the Kronecker delta functions in equations 3.16 to 3.19.

It is known that for wave motions on an infinite plate, wave motions with supersonic phase velocities appear as a damping of the plate response as energy can be transferred from the plate to the fluid becoming acoustic radiation, whereas wave motions with subsonic phase velocities lead to a virtual mass term as no energy is transferred. This observation is also true for finite plates, the additional constraint of the edges of the plate being sufficient to obtain acoustic radiation from subsonic wave motions on the plate. The volume-velocity cancellations that lead to radiation from these subsonic wave motions have been described above, following Maidanik[2]. The essential features of supersonic and subsonic waves on the plate are still preserved, however, as shown by the large radiation coefficient of acousticallyfast modes (equation 3.5) and the large modal inertia coefficient of acoustically slow modes (equation 3.20).

The forms of the coupling coefficients suggest that the set of equations (2.5) can be conveniently rewritten in the form

$$B_{mn} V_{mn} - i\omega \sum_{qr}{}' S_{mnqr} V_{qr} + \omega \sum_{qr}{}' T_{mnqr} V_{qr} = -\omega F_{mn} \qquad (3.21)$$

where now the term $\rho_o c_o \frac{k_o}{k_{mn}}$ of equation (3.20) is included in the term B_{mn}. In what follows, T_{mnqr} will be defined solely as in equations (3.16) to (3.19). B_{mn} is defined as

$$B_{mn} = D k_{mn}^4 - \omega^2 M_p - i\omega^2 m_p \eta_{mn} \operatorname{sgn} \omega$$

with

$$M_p = \begin{cases} m_p\left(1 + \dfrac{\rho_0}{m_p k_{mn}}\right), & k_{mn} > k_0 \\ m_p, & k_{mn} < k_0 \end{cases}$$

The solution of the set of equations (3.21) is discussed in the following sections. The discussion is restricted to frequencies below the acoustic critical frequency.

4. <u>The Low Frequency Limit</u>

At frequencies such that $k_o\ell_1$, $k_o\ell_3 < \pi$ all modes are of corner mode radiation character. The radiation and coupling coefficients are described by equations (3.6) and (3.19). Each of the modal equations (3.21) is thus of similar form. They can be written:

$$B_{mn} \; [illegible] \; \sum_{q} S_{mnqr} \; [illegible] \; \sum_{r} T^{1}_{mnmr} \; [illegible] \; \sum_{q} T^{3}_{mnqn} \; [illegible] = [illegible] \; \tilde{p}_{mn}$$

S_{mnqr} is described by equation (3.6). T^1_{mnmr} and T^3_{mnqn} are defined

by the expressions

$$T_{mnmr} = - \frac{4\rho_0 c_0}{\pi A_p} \, k_0 \ell_1 \, \frac{k_n k_r}{k_{mn}^2 \, k_{mr}^2} ,$$

and

$$T^{s}_{mnqn} = - \frac{4\rho_0 c_0}{\pi A_p} \, k_0 \ell_3 \, \frac{k_m k_q}{k_{mn}^2 \, k_{qn}^2} .$$

The summations in equations (4.1) are over all (q,r) modes with the same parity as (m,n). Thus, there are four separate sets of equations (4.1) to be solved corresponding to the four combinations of even and odd values of m and n. We will, of course, assume that the nomenclature of equation (4.1) covers all four such cases.

The set of equations (4.1) cannot be solved exactly. We look for an approximate solution of these equations which includes the main features of the coupling. The solution for v_{mn} is obviously a function of all the modal forcing functions p_{mn}. However, we might reasonably expect that the solution for v_{mn} can be written in the form of a modal admittance function Y_{mn} multiplying the $(m,n)^{th}$ modal forcing function p_{mn}, with, in addition higher order terms of smaller magnitude involving the coupling coefficients and all the forcing functions p_{mn}. We can obtain such a solution: the interpretation of the approximations

we make in doing so is more fully discussed later in this section.

The equations (4.1) are first rewritten in the form

$$B_{mn} v_{mn} + \omega \sum_r^{1} T^1_{mnmr} v_{mr} + \omega \sum_q^{1} T^3_{mnqn} v_{qn} = -i\omega \hat{f}_{mn} \qquad 4.2$$

where

$$\hat{f}_{mn} = P_{mn} - \sum_{q,r}^{1} S_{mnqr} v_{qr} \quad .$$

The set of equations (4.2) can be rearranged to give v_{mn} in terms of the new force $\hat{f}_{mn}$. The term

$$\omega \sum^{1} T^1_{mnmr} v_{mr}$$

is evaluated by substituting for v_{mr} from the equations (4.2). We obtain

$$\omega \sum_r^{1} T^1_{mnmr} v_{mr} = \omega \sum_r^{1} \frac{T^1_{mnmr}}{B_{mr}} \left\{ -i\omega \hat{f}_{mr} - \omega \sum_q^{1} T^3_{mrqr} v_{qr} \right\}$$

$$- \omega^2 \sum_r^{1} \frac{T^1_{mnmr}}{B_{mr}} \sum_s^{1} T^1_{mrms} v_{ms} \quad .$$

where we have used the relationship

$$T^{1}_{mnmr}\,T^{1}_{mrmv} = T^{1}_{mrmr}\,T^{1}_{mnmv} \, . \qquad 4.4$$

[It may be helpful to note here that the sufficies m, q, u will always refer to mode numbers in the x_1 direction; the sufficies n, r, v will always refer to mode numbers in the x_3 direction.]

If we now substitute from equation (4.3) into equation (4.2) we obtain the exact form:

$$B^{1}_{mn}\,v_{mn} + \omega \sum_{q} T^{3}_{mnqn}\,v_{qn} = -i\omega\, g_{mn} \qquad 4.5$$

where

$$g_{mn} = f_{mn} + \omega \sum_{r} \left\{ \frac{T^{1}_{mrmr}}{B_{mr}} f_{mn} - \frac{T^{1}_{mnmr}}{B_{mr}} f_{mr} \right\} + i\omega \sum_{q,r} \left\{ \frac{T^{1}_{mrmr}}{B_{mr}} T^{3}_{mnqn}\,v_{qn} - \frac{T^{1}_{mnmr}}{B_{mr}} T^{3}_{mrqr}\,v_{qr} \right\}$$

and B^{1}_{mn} is defined by the expression

$$B^{1}_{mn} = B_{mn} \left(1 + \omega \sum_{r} \frac{T^{1}_{mrmr}}{B_{mr}} \right) \, .$$

We thus find, already, an admittance function containing at least some of the T^1_{mrmn} type coupling effects.

We now repeat the above process. The term

$$\omega \sum_q' T^3_{mnqn} v_{qn}$$

is evaluated using the equations (4.5). When we substitute for this term back into equation (4.5), noting the relation:

$$T^3_{mnqn} T^3_{qnun} = T^3_{qnqn} T^3_{mnun} ,$$

we obtain the exact form:

$$B^{13}_{mn} v_{mn} - i\omega \sum_{q,r}' S_{mnqr} v_{qr} = -i\omega h_{mn} , \qquad 4.6$$

where

$$h_{mn} = p_{mn} + \omega g_{mn} \sum_q' \frac{T^3_{qnqn}}{B^1_{qn}} - \omega \sum_q' \frac{T^3_{mnqn}}{B^1_{qn}} g_{qn} ,$$

and B^{13} is defined by the expression

$$B^{13}_{mn} = B_{mn}\left(1 + w\sum_r \frac{T^1_{mrmr}}{B_{mr}} + w\sum_q \frac{T^3_{qnqn}}{B_{qn}}\right).$$

Once again we can solve the equations (4.6) as S_{mnqr} is symmetric and satisfies the relation

$$S_{mnqr}\,S_{qruv} = S_{mnuv}\,S_{qrqr}.$$

We finally obtain the expression

$$\chi_{mn}\,v_{mn} = -iw\,h_{mn} + iw^2\sum_{q,r} h_{mn}\frac{S_{qrqr}}{B^{13}_{qr}} - iw^2\sum_{q,r} h_{qr}\frac{S_{mnqr}}{B^{13}_{qr}}, \qquad 4.7$$

where

$$\chi_{mn} = B_{mn}\left\{1 - iw\sum_{q,r}\frac{S_{qrqr}}{B_{qr}} + w\sum_r\frac{T^1_{mrmr}}{B_{mr}} + w\sum_q\frac{T^3_{qnqn}}{B_{qn}}\right\}. \qquad 4.8$$

The equations (4.7) are an exact relation derived from equations (4.1). However, they do not represent a solution of the set of equations (4.1): there are still terms in v_{mn} on the right hand side. We could, in principle, perhaps, continue "solving" the set of equations as above. However, the modal velocity terms we would now deal with are second order, that is, they are multiplied by products of the form $S_{mnqr}\,T^{1}_{mnmv}$. We would thus be dealing with second order coupling effects. The first order coupling effects can be extracted directly from equation (4.7). The admittance function Y_{mn} we assumed could be found is seen to be

$$Y_{mn} = -\frac{i\omega}{X_{mn}} .$$

The remaining terms in equation (4.7) include both the excitation of the modal velocity v_{mn} by all the modal forcing functions, and also the higher order coupling effects.

The simplest way to extract the dominant terms is by redoing the above analysis and discarding the higher order terms as the analysis proceeds. Thus, in equation (4.5) we may write for g_{mn} the approximate form

$$g_{mn} = f_{mn} + \omega\, p_{mn} \sum_{r} \frac{T^{1}_{mrmr}}{B_{mr}} - \omega \sum_{r} \frac{T^{1}_{mnmr}}{B_{mr}}\, p_{mr} .$$

The second order terms we neglect here are

$$\omega \sum_{q,r}{}' \sum_{v}{}' \left\{ T'_{mvmv} S_{mnqr} - T'_{mnmv} S_{mvqr} \right\} \frac{v_{qr}}{B_{mv}} \qquad 4.9$$

From the definitions of the coupling terms we find that

$$T'_{mvmv} S_{mnqr} - T'_{mnmv} S_{mvqr} \propto \frac{k_v}{k_{mv}^2 k_n} - \frac{k_n}{k_{mn}^2 k_v} . \qquad 4.10$$

Later, when evaluating the magnitude of the coupling we will find it sufficient to consider only the coupling between resonant modes. We would then expect that for resonant modes k_v and k_n in equation (4.10) are of similar magnitude. The terms we neglect, (4.9), already second order, thus also tend to cancel each other.

By neglecting other similar product terms we can finally obtain an approximate form of equation (4.7). We will consider this to be the required approximate solution of the set of equations (4.1). This solution is

$$\begin{aligned} X_{mn} v_{mn} &= -i\omega p_{mn} + (i\omega)^2 \sum_{qr}{}' \frac{R_{qrqr}}{B_{qr}} p_{mn} \\ &\quad - (i\omega)^2 \sum_{q,r}{}' \frac{R_{mnqr}}{B_{qr}} p_{qr} \end{aligned} \qquad 4.11$$

where

$$R_{mnqr} = S_{mnqr} + i\, T^{1}_{mnmr}\, \delta_{mq} + i\, T^{3}_{mnqn}\, \delta_{nr} \,, \qquad 4.12$$

and χ_{mn} is defined by equation (4.8). We note again that the term in expression 3.20 is not included in the definition of R_{mnmn}.

Our approximate solution is in terms of an admittance $Y_{mn} = -i\omega\psi_{mn}$ which itself includes both modal radiation and modal inertia coupling effects. We have also retained the first order forces on the $(m,n)^{th}$ mode of the other modal forcing functions. We note that equation (4.11) can be rewritten in the form

$$v_{mn} = -i\omega \frac{p_{mn}}{B_{mn}} - \frac{(i\omega)^2}{\chi_{mn}} \sum_{q,r} \frac{R_{mnqr}}{B_{qr}}\, p_{qr} \,. \qquad 4.13$$

However, the importance of ψ_{mn} in this equation is obscured. We will find when considering power flow into and out of the plate that equation (4.11) is a more useful form. The summations in equation (4.11) are concerned with the forces on the (m,n) mode due to the modal components p_{qr}, and the forces exerted by p_{mn} on all the other modes. As far as the total velocity field of the plate is concerned these combined effects will tend to be self cancelling. We then find that the important coupling effects are contained in ψ_{mn}. This becomes evident when we consider the power flow in the system.

We also note at this point that the solution (4.7) can be obtained directly by evaluating

$$i\omega \sum_{q,r} R_{mnqr} v_{qr}$$

where R_{mnqr} is defined by equation (4.12). The form of the admittance function does not become as easily evident in this case, and little is saved in the amount of algebra required as we must still expand R_{mnqr} in terms of S_{mnqr}, T^1_{mnmr} and T^3_{mnqn}. Having obtained the quasi-solution (4.7), however, we recognize that the approximate solution (4.11) follows by assuming that

$$R_{mnqr} R_{qruv} = R_{qrqr} R_{mnuv} \quad . \qquad (4.14)$$

The error terms in this last expression are just those terms, such as (4.10), that we discarded to obtain our approximate solution (4.11). In some of the analysis that follows we will make direct use of the approximate relation (4.14).

Having obtained an approximate solution for the modal velocity field of the plate, we now use this solution to obtain estimates of the total power input to the system and the radiated acoustic power. For simplicity in obtaining these results some assumptions will be made about the nature of the forcing field. We will assume that it is both spatially homogeneous in the plane of the plate and is temporally stationary, that is, its correlation function is a function of both spatial and tem-

poral separation only. We further assume that the characteristic correlation lengths, including the convected length, if any, in which the field decays, are much smaller than the dimensions of the plate. These last criteria are those of Dyer[11]. These assumptions are not unreasonable and correspond well to typical physical situations. They are useful in that they lead to no coupling of the plate modes by the forcing field. It has already been noted that we assume no interaction between the external force and either the plate vibration or the associated acoustic field, i.e., the external force is specified irrespective of the response of the system.

We can express the correlation of two modal forces as follows:

$$\langle p_{mn} p_{qr}^* \rangle = \iint_{A_p A_p} \langle p(\underline{x},\omega)\, p^*(\underline{\xi},\omega)\, \psi_{mn}(\underline{x})\, \psi_{qr}(\underline{\xi}) \rangle \, d\underline{x}\, d\underline{\xi}$$

where <X> denotes the expected value of X. Because of our assumption of homogeneity we may write

$$\langle p_{mn} p_{qr}^* \rangle = \frac{1}{(2\pi)^2} \iint_{A_p A_p} \int_{-\infty}^{\infty} \phi(\underline{k},\omega)\, e^{i\underline{k}\cdot(\underline{x}-\underline{\xi})}\, \psi_{mn}(\underline{x})\, \psi_{qr}(\underline{\xi})\, d\underline{x}\, d\underline{\xi}\, d\underline{k}$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \phi(\underline{k},\omega)\, S_{mn}(\underline{k})\, S_{qr}^*(\underline{k})\, d\underline{k} \quad ,$$

where $\phi(\underline{k},\omega)$ is the wavenumber-frequency Fourier transform of the correlation function of the force $p(\underline{x},t)$. Dyer's[11] criteria imply that $\phi(\underline{k},\omega)$ is essentially constant in $\underline{k}$. The orthogonality of the characteristic functions then leads to the simple result

$$\langle p_{mn}\, p^*_{qr} \rangle = \phi_{mn}\, \delta_{mq}\, \delta_{nr} \qquad \text{4.15}$$

where

$$\phi_{mn}(\omega) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \phi(\underline{k},\omega) \left| S_{mn}(\underline{k}) \right|^2 d\underline{k}$$

is a modal correlation function.

The correlation functions $\langle p_{mn}\, p^*_{qr} \rangle$ are essentially Powell's[8] joint acceptances. When Dyer's criteria are not satisfied we must include all these functions in the analysis. However, as these criteria are satisfied in a large range of physical situations it is assumed in the analysis that follows that the joint acceptances always satisfy the simple relation (4.15). We may, indeed, reasonably conjecture that the two modal coupling mechanisms, the external field and the acoustic field, act independently, at least to first order. Thus, as far as a study of fluid loading is concerned there is no loss of generality in assuming that Dyer's criteria are met. We expect the extension of the analysis to include the additional coupling effects

to be straightforward, although, possibly, laborious. An approximate solution is suggested in an appendix for cases when Dyer's criteria are not satisfied.

We now estimate the energy flow into and out of the plate. We consider first the radiated acoustic power. The total power flow into the fluid from the plate is defined to be an integral over the area of the plate, namely

$$\int_{A_p} \langle P(\underline{x}, 0, t)\, v(\underline{x}, t) \rangle \, d\underline{x} \quad .$$

For the present we will not restrict attention to the real part of this expression but rather consider the total complex power. We may refer to the real and imaginary components as the resistive power and reactive power, respectively. The resistive power describes the drain of energy by radiation to infinity; the reactive power is a measure of the energy that is stored in the fluid. To obtain the spectral density of the radiated power, $H(\omega)$, we require the expected value of $P(\underline{x}, 0, t)$ times $v(\underline{x}, t + \tau)$ for all time delays τ. We again consider both the real and imaginary parts of $H(\omega)$.

$$H(\omega) = \int_{-\infty}^{\infty} \int_{A_p} \langle P(\underline{x}, 0, t)\, v(\underline{x}, t+\tau) \rangle \, e^{i\omega\tau} \, d\underline{x} \, d\tau \quad .$$

By using the modal expressions and the orthogonality of the characteristic functions we obtain the result

$$H(\omega) = \sum_{m,n} < P_{mn}(\omega)\, v^{*}_{mn}(\omega) > \qquad (4.16)$$

where the summation is over all modes. P_{mn} in our present notation is given by the expression

$$P_{mn}(\omega) = \sum_{q,r} R_{mnqr}\, v_{qr} - i \rho_0 c_0 \frac{k_0}{k_{mn}}\, v_{mn} ,$$

where R_{mnqr} is defined by equation (4.12) and we have included the term T_{mnmn} separately. If we substitute for v_{qr} using equation (4.11) we obtain directly

$$P_{mn} = i\omega \sum_{qr} \frac{R_{mnqr}}{\lambda_{qr}} P_{qr} - i \rho_0 c_0 \frac{k_0}{k_{mn}}\, v_{mn}$$

$$- (i\omega)^2 \sum_{qr} \sum_{st} \left\{ \frac{R_{mnqr}\, R_{qrst}\, P_{qr}}{\lambda_{qr}\, B_{st}} - \frac{R_{mnqr}\, S_{qrst}\, P_{st}}{\lambda_{qr}\, B_{st}} \right\}$$

We now use the fact that our solution is based on the approximate relation (4.14). To this degree of approximation the double summation above can be neglected. A similar double summation is neglected when we

substitute for P_{mn} and v^{*}_{mn} in equation (4.16). When Dyer's criteria are assumed to hold, the expression for the (complex) radiated power reduces to the simple form

$$\Pi(\omega) = \sum_{m,n} S_{mnmn} \left| \frac{\omega}{Z_{mn}} \right|^2 \phi_{mn} - i \rho_0 c_0 \sum_{m,n} \frac{k_0}{k_{mn}} \langle |v_{mn}|^2 \rangle \qquad (4.17)$$

The real part of this expression, the spectral density of the power radiated to infinity, say $\Pi(\omega)$, is seen to be

$$\Pi(\omega) = \rho_0 c_0 \sum_{m,n} \sigma_{mn} \left| Y_{mn} \right|^2 \phi_{mn} \qquad (4.18)$$

where, following Leehey[3], we have written $S_{mnmn} = \rho_0 c_0 \sigma_{mn}$, σ_{mn} being the modal radiation efficiency ($\sigma_{mn} = 1$ for acoustically fast modes). Equation (4.18) is similar in form to previously published results for the radiated power obtained either by neglecting fluid loading effects or by assuming light fluid loading. The importance of the admittance function is now clearly seen. The fluid loading effects, the modal coupling induced both by radiation loading and inertia loading, are all included in the admittance functions Y_{mn}.

For a specified external force acting on the plate the spectral density of acoustic power radiated by the plate can be computed directly

from equation (4.18). When the plate modal density in frequency space is high, more useful expressions can be obtained by averaging $\Pi(\omega)$ over narrow bands of frequency and considering only the power radiated by modes with resonanace frequencies within the frequency band. It is assumed that even with the added damping due to fluid loading the resonance peaks are sufficiently marked that the radiated power is indeed mainly from the resonant modes. It is again emphasized that the resonance frequencies referred to are merely the frequencies at which the real components of the admittance functions Y_{mn} vanish. The inertia terms, of course, cause a decrease this frequency from the in vacuo resonance frequency. By averaging over a narrow band of frequencies, $\Delta\omega$, equation (4.18) for the radiated power spectral density can be written

$$\Pi(\omega)\Delta\omega = \rho_o c_o \sum_{m,n} \sigma_{mn}(\omega)\, \Phi_{mn}(\omega) \int_{\omega-\frac{\Delta\omega}{2}}^{\omega+\frac{\Delta\omega}{2}} \left|Y_{mn}(\omega)\right|^2 d\omega \qquad 4.19$$

where the summation is over those modes with resonance frequencies in the $\Delta\omega$ band.

The integration over $|Y_{mn}|^2$ will now be carried out. To emphasize the coupling effect we write Y_{mn} in the form

$$Y_{mn} = -i\omega \left\{ B_{mn} \cdot i\omega \rho_o c_o \sum_{q,r} \frac{B_{mn}}{B_{qr}} \left(\sigma_{qr} + i\tau_{qr}\right)\right\}^{-1} \qquad 4.20$$

where T_{qrqr} is replaced by $\rho_o c_o \tau_{qr}$. The terms $\rho_o c_o \tau_{qr}$ are, in general, con-

siderably smaller than the inertia term $\rho_o c_o \frac{k_o}{k_{mn}}$ included in B_{mn}. The effect of inertial coupling will thus, presumably, be small. When averaging, as in equation (4.19), we are interested in those modes with resonance frequencies in $\Delta\omega$. Conversely, when estimating the coupling, as in equation (4.20) we are interested in frequencies near the resonance frequency of Y_{mn}, and hence, as τ_{qr} is small, near the resonance frequency of B_{mn}. The added damping is seen to be due to σ_{mn}, as in light fluid loading approximations, plus other modal radiation coefficients σ_{qr} modified by the terms $\frac{B_{mn}}{B_{qr}}$. To evaluate the expression (4.20), the frequency is written in terms of the resonance frequency ω_{mn} of B_{mn},

$$\omega = \omega_{mn}(1+\varepsilon) \qquad 4.21$$

where ε corresponds to the small variations of frequency within the $\Delta\omega$ band. The resonance frequency of B_{qr} is related to that of B_{mn} by the expression

$$\omega_{mn} = \omega_{qr}(1+\lambda_{qr}^{mn}) \qquad 4.22$$

where λ_{qr}^{mn} is a measure of the separation of the resonance peaks. At frequencies within $\Delta\omega$ we can now write

$$\begin{aligned} B_{mn} &= D k_{mn}^4 - \omega_{mn}^2(1+\varepsilon)^2 M_p - i\omega_{mn}^2(1+\varepsilon)^2 m_p \eta_{mn} \\ &\approx -2\omega_{mn}^2 M_p\left(\varepsilon + i\frac{m_p}{M_p}\frac{\eta_{mn}}{2}\right) \end{aligned} \qquad 4.23$$

and, similarly,

$$B_{qr} \approx -2\,\omega_{qr}^2 M_p \left(\varepsilon + \lambda_{qr}^{mn} + i \frac{m_p}{M_p} \frac{\eta_{qr}}{2} \right) \qquad 4.24$$

The summation in equation (4.20) can now be expressed in the form

$$\sum_{q,r} \frac{B_{mn}}{B_{qr}} (\sigma_{qr} + i\tau_{qr}) = \sum_{q,r} (\sigma_{qr} + i\tau_{qr}) \frac{\left(\varepsilon + \frac{i m_p}{2 M_p} \eta_{mn} \right)\left(\varepsilon + \lambda_{qr}^{mn} - \frac{i m_p}{2 M_p} \eta_{qr} \right)}{\left(\varepsilon + \lambda_{qr}^{mn} \right)^2 + \left(\frac{m_p \eta_{qr}}{M_p\, 2} \right)^2} \qquad 4.25$$

It is clearly seen that the important contributions to the summation at the resonance frequency of B_{mn} come from those modes with resonance frequencies close to ω_{mn}. Thus, the amount of coupling is controlled by the spacing of the resonance frequencies, there being a large coupling effect into those modes with $\lambda_{qr}^{mn} << \eta_{mn}$, that is, when the separation is less than the effective widths of the resonance peaks. On the other hand, since we assume that the modes are only appreciably excited at their resonance frequencies, we neglect the coupling between modes whose resonance peaks do not overlap.

Two cases must be distinguished in evaluating expression (4.25). When the structural damping is small, that is when $\lambda_{qr}^{mn} >> \eta_{mn}$ for all

modes, then there is no effective modal coupling, and expression (4.25) reduces to a single term. We obtain directly in this case that at frequencies near resonance:

$$Y_{mn}(\omega) = \frac{-i}{2\omega M_p}\left\{\varepsilon + \frac{i m_p}{2M_p}\left(\eta_{mn} + \frac{\rho_o c_o}{\omega m_p}\sigma_{mn}\right)\right\}^{-1} \qquad 4.26$$

where we have dropped the term $\rho_o c_o \tau_{mn}$ as it is much smaller than the term $\rho_o c_o \frac{k_o}{k_{mn}}$. ω_{mn} is, of course, already defined with the inertia term $\rho_o c_o \frac{k_o}{k_{mn}}$ included. Expression (4.26) is not strictly complete as we have not included the possibility of two modes having the same resonance frequency, in particular as in the degenerate case of a square plate. However, at very low values of the structural damping we see that we may reasonably neglect modal coupling. The particular case of a square plate can be included in this analysis simply by doubling the value of the radiation damping.

The second case concerns plates with large structural damping and high modal densities. Considerable modal interaction now occurs. To evaluate the coupling effects we may suppose the modes evenly spaced in frequency, in which case the separation between adjacent modes is obtained from equation (4.22) as

$$\lambda = \frac{1}{\omega \nu(\omega)} ,$$

where ν is the frequency modal density. We are now interested in the case $\lambda >> \eta_{mn}$, that is, we have the condition

$$\omega \eta_{mn} \nu >> 1 \qquad 4.27$$

The summation (4.25) can be treated approximately as an integral in λ in the form

$$\cdot^{\,1}_{q,r} = \frac{\omega\nu}{4}\left\{<\sigma_{mn}> + i<\tau_{mn}>\right\}\int_{-\infty}^{\infty} \frac{\left(\varepsilon + i\frac{m_p}{M_p}\frac{\eta_{mn}}{2}\right)\left(\varepsilon+\lambda - i\frac{m_p}{M_p}\frac{\eta_{qr}}{2}\right)}{(\varepsilon+\lambda)^2 + \left(\frac{m_p}{M_p}\frac{\eta_{qr}}{2}\right)^2}\, d\lambda$$

$$= \frac{\omega\nu}{4}\left\{<\sigma_{mn}> + i<\tau_{mn}>\right\}\frac{m_p}{M_p}\,\eta_{mn}\,\frac{\pi}{8}$$

The limits have been taken as $\pm\infty$ to include all modes, a factor 1/4 being introduced to account for the fact that there is coupling only between modes of the same parity. The imaginary part of the integral involves the factor $(\eta_{mn} - \eta_{qr})$ and has been neglected. We thus obtain for the admittance function near the resonance frequency

$$Y_{mn} = \frac{i}{2\omega M_p}\left\{\varepsilon + i\frac{m_p}{2M_p}\left(\eta_{mn} + \frac{\pi}{4}(\omega\nu\eta_{mn})\frac{\rho_0 c_0}{\omega M_p}<\sigma_{mn}>\right)\right\}^{-1}. \qquad 4.28$$

Again we have dropped the additional inertia terms, T_{mn}. Had we included these terms, the effective mass of the plate would be described by

$$m_p \left\{ 1 + \frac{\rho_o c_o}{\omega m_p} \frac{k_o}{k_{mn}} + \frac{\rho_o c_o}{\omega M_p} \frac{(\omega \gamma \eta_{mn}}{2 k_{mn}^4 A_p} < \left(k_o l_1 k_n^2 + k_o l_3 k_m^2 \right) > \right\}$$

Typically we expect that the last term here is small, but, if necessary, it can be included in the estimate of the resonance frequency ω_{mn}. Equation (4.28) still correctly describes the nature of the function Y_{mn} near its resonance frequency although this resonance frequency is now slightly changed. The important fluid loading inertia term, the modal "self" inertia $\frac{\rho_o c_o}{\omega} \frac{k_o}{k_{mn}}$, has already been included.

The dominant fluid loading effects are the change of resonance frequency by the modal "self" inertia and the, possibly, large increase in the total damping. Some typical magnitudes of these effects are discussed in section 7. The inertia modal coupling terms, although adding considerable complexity to the analysis, are found to have little effect on the admittance of the plate-fluid system. In this context it is worth noting that if we neglect the terms T^1_{mnmr} and T^3_{mnqr} throughout the analysis, equations (4.8), (4.11) and (4.12) then together represent an exact solution of the set of modal equations (4.1).

The integration over $|Y_{mn}|^2$ (now as a function of ϵ) can now be performed as in previous analyses (e.g. references 3 and 4) by assuming

that the total damping is sufficiently small that the effective resonant bandwidth is much less than the bandwidth $\Delta\omega$. We obtain two estimates of the radiated power spectral density by using expressions (4.26) and (4.28), respectively:

i) $\omega \upsilon \eta_{mn} \ll 1$

$$\Pi(\omega)\,\Delta\omega = \frac{\pi}{2M_p} \sum_{m,n} \phi_{mn} \frac{\rho_0 c_0 \sigma_{mn}}{\omega m_p \eta_{mn} + \rho_0 c_0 \sigma_{mn}}, \qquad 4.29$$

ii) $\omega \upsilon \eta_{mn} \gg 1$

$$\Pi(\omega)\,\Delta\omega = \frac{\pi}{2M_p} \sum_{m,n} \phi_{mn} \frac{\rho_0 c_0 \sigma'_{mn}}{\omega m_p \eta_{mn} + \frac{\pi}{4}(\omega \upsilon \eta_{mn}) \frac{m_p}{M_p} \rho_0 c_0 \langle \sigma_{mn} \rangle}. \qquad 4.30$$

The summations are over all the resonant modes in the $\Delta\omega$ frequency band.

In a similar way we can obtain an estimate of the total input power to the plate-fluid system. Thus, corresponding to equation (4.16):

$$\Pi_i(\omega) = \sum_{m,n} \langle p_{mn} \upsilon^*_{mn} \rangle$$

$$= \sum_{m,n} \phi_{mn}(\omega) \frac{i\omega}{B^*_{mn}} \left(1 - i\omega Y^*_{mn} R^*_{mnmn}\right)$$

The required power is the real part. This expression can also be estimated by averaging over the $\Delta\omega$ frequency band. The integral over the second term is zero. We obtain an expression for the spectral density of input power which is independent of either the structural or the radiation damping: the only fluid loading effect is the change of modal resonance frequency by the modal self inertia term. Thus,

$$\Pi_i(\omega)\,\Delta\omega = \frac{\pi}{2M_p}\sum_{m,n}{}' \phi_{mn}(\omega) = \frac{\pi}{2M_p}\,\nu\,\Delta\omega < \phi_{mn} >$$

It is shown in reference 4 that the modal correlation functions $\phi_{mn}(\omega)$ are the same for all resonant modes in a narrow frequency band if the correlation lengths of the forcing field are considerably less than the corresponding modal wavelengths. For excitation by a turbulent boundary layer this condition is satisfied at frequencies well above the hydrodynamic critical frequency, that is, the resonance frequency of the plate modes with wavespeed on the plate equal to the convection velocity of the forcing field. In this case, the functions ϕ_{mn} can be taken outside the summations in equations (4.24) and (4.25). The radiated power spectral density is then linearly related to the input power spectral density.

A comparison of some typical numerical values of the radiated power spectral density showing the magnitude of the fluid loading terms is made in section 7.

5. The High Frequency Limit

At high frequencies, $k_o\ell_1$, $k_o\ell_3 >> \pi$, the radiation and coupling characteristics of the modes are not the same for all modes as in the low frequency case. The coupled modal equations are thus considerably more complex and correspondingly more difficult to solve. In this section an approximate solution of the modal equations we have obtained is discussed. We start by estimating the magnitude of the force exerted on a mode due to its being coupled to other types of modes. Thus, for example, we make an estimate of the total effect on the response of an edge mode of the coupling into all the corner modes, based on a typical corner mode resonant amplitude and the total number of resonant coupled corner modes. This approach leads to a first approximation in which the corner and edge mode responses can be solved for separately, including only the coupling into other corner and edge modes, respectively. We include the nonresonant acoustically fast modes in the analysis and calculate their response due to coupling to resonant edge modes. This step seemed necessary as the associated radiation from these highly efficiently radiating modes could be of importance. An estimate of the total radiated power shows, however, that the effect of these modes is negligible. From some numerical estimates of the radiated power made in section 7 it would also seem that the modal component of the force exciting the nonresonant acoustically fast modes also leads to a negligibly small contribution to the power.

The corner mode contribution to the radiated power is included in this high frequency analysis. It is known that under light fluid loading conditions at frequencies where edge mode radiation occurs, this radiation dominates that from corner modes. Under dense fluid loading conditions, however, the higher radiating efficiency of the edge modes causes correspondingly higher damping of the modes, the effect being such that, in some cases, the edge mode radiation dominance does not occur except at frequencies where very many edge modes are excited. Structural damping is important in determining the relative magnitudes of the edge and corner mode radiation. This is discussed in section 7.

We first consider the effect on the edge mode response of the coupling into other types of modes. The complete modal equation for an X-type edge mode response will be written

[illegible]

Here, v^{X}_{mn} is the modal velocity amplitude of an X-type edge mode, R^{XF}_{mnqr} is the coupling coefficient between the (m,n) X-type edge mode and the (q,r) acoustically fast mode: similar interpretations follow obviously for the other notations, C indicating a corner mode. The

summations are over all the coupled modes taking the parity of the mode numbers into account. Similar equations exist for Y-type edge modes, corner modes and acoustically fast modes.

Estimates of the magnitudes of the summations inequation (5.1) are made using the values of the coupling coefficients obtained in section 3. Thus, using equations (3.7) and (3.16),

[illegible]

using equations (3.8) and (3.17),

[illegible]

and using equations (3.9) and (3.18)

[illegible]

Now, if there is a considerable amount of modal interaction, the relative importance of the terms in expressions (5.2), (5.3) and (5.4) can be estimated by assuming that all the acoustically slow modes in a narrow wavenumber band Δk are coupled together. We also include all the nonresonant acoustically fast modes. The case of low structural damping with essentially no modal interaction can then be obtained as a particular case. We assume when estimating the relative importance of the various terms that all the resonant acoustically slow modes and all the non-resonant acoustically fast modes have the same velocity amplitude, respectively. The radiation coupling and the inertia coupling are estimated separately. The following ratios are obtained from equations (5.2), (5.3) and (5.4):

i) radiation coupling force on X-type edge mode

(5.5)

ii) inertia coupling force on X-type edge mode

We cannot yet estimate the effect of the acoustically fast modes on the edge mode response without first considering the acoustically fast mode equation. This is:

$$\left(\omega^2 m_p + i\omega\rho_o c_o\right) v_{qr}^F + i\omega \sum_{m,n} R_{qrmn}^{FF} v_{mn}^F$$
$$+ i\omega \sum_{m,n} R_{qrmn}^{FX} v_{mn}^X + i\omega \sum_{m,n} R_{qrmn}^{FY} v_{mn}^Y = i\omega p_{qr}^F \qquad 5.7$$

The acoustically fast modes we treat are at frequencies well above their resonance frequencies: they are thus mass controlled. This is reflected in the term $(\omega^2 m_p + i\omega\rho_o c_o)$ which is merely the mass controlled admittance plus the modal radiation term $\omega\rho_o c_o$. There is no virtual mass term for these modes. The summations in equation (5.7) can be estimated as before. However, the dominant term in this equation is the modal admittance: a comparison of the terms shows that v_{qr}^F is at least a factor $(k_o \ell)^{-1}$ times v_{mn}^X. That the nonresonant amplitudes v_{qr}^F are very much smaller than the resonant amplitudes of v_{mn}^X is hardly a surprising fact. But it makes possible a number of approximations. In equation (5.7), the summation $\overset{F}{\Sigma}$ is dropped. Only the inertia coupling terms are retained in $\overset{X}{\Sigma}$ and $\overset{Y}{\Sigma}$: the radiation coupling terms are negligible compared with the radiation term $\omega\rho_o c_o$. The edge mode inertia coupling is retained as being of possible importance, particularly if the modal force p_{qr}^F is small. Returning to equations (5.5)

and (5.6), we now find that where there is considerable modal interaction, so that Δk is reasonably large, the edge mode [illegible]mations dominate, although not to any great extent. For example, if $k_\eta = k_o/2$ and $k_{mn} = 10\ k_o$, the ratios of edge to corner mode summations in equations (5.5) and (5.6) are, respectively $1 : 1/10$ and $1 : k_o/1200\ \Delta k$. It is thus reasonable to include only the edge mode summation in the edge mode equation (5.1). (As we are considering frequencies such that $k_o \ell_1$, $k_o \ell_3 >> \pi$ the acoustically fast mode summations are obviously negligible.) In a similar way we find that the dominant coupling in the modal equation for the corner mode response is into other corner modes. We have thus reduced the problem at high frequencies to solving for the edge and corner mode responses separately, neglecting the coupling between edge and corner modes. Having found the edge mode response we can then estimate the nonresonant acoustically fast mode response. The acoustically fast modes are retained in the analysis until we can show when estimating the radiated power that their effect is, in general, negligible.

The reduced s[illegible]al equations is as follows

$$B^{x}_{mn}\, v^{x}_{mn} - i\omega \sum_{r} \left(S^{x\ x}_{mn\,mr} + i\, T^{x\ x}_{mn\,mr} \right) v^{x}_{mr} = -i\omega\, p^{x}_{mn} \qquad 5.8$$

where only the first term of equation (3.16) is now included, the other being negligible as shown by equation (5.6); with a similar equation

for Y-type edge modes:

$$B^{c}_{mn} v^{c}_{mn} - i\omega \sum_{q,r} \left(S^{c\ c}_{mnqr} + i\, T^{c\ c}_{mnqr} \right) v^{c}_{qr} = - i\omega\, p^{c}_{mn} \qquad 5.9$$

where $S^{c\ c}_{mnqr}$ and $T^{c\ c}_{mnqr}$ are defined by equations (3.6) and (3.19) respectively: and

$$\left(\omega^2 m_p + i\omega \rho_0 c_0 \right) v^{F}_{mn} - \omega \sum_{r}^{1} T^{F\ X}_{mnmr}\, v^{X}_{mr} - \omega \sum_{q}^{1} T^{F\ Y}_{mnqn}\, v^{Y}_{qn} = i\omega\, p^{F}_{mn} \qquad 5.10$$

The dominant terms in these reduced equations have been obtained from estimates based on the numbers of resonant modes. It is easier now to solve the equations as they stand considering all the modes in the summations. The resonant modes will then again be picked out when we subsequently average over a narrow band of frequencies.

Equation (5.9) is almost identical to equation (4.1), the only difference being that now as not all modes are corner modes, the summations in equation (5.10) are over only a limited number of modes. The approximate solution is similar to equation (4.11) with this restriction applied to the summations.

The coupling terms in equation (5.8) satisfy the relation

$$R^{X\ X}_{mnmr}\, R^{X\ X}_{mrmv} = R^{X\ X}_{mnmv}\, R^{X\ X}_{mrmr}$$

exactly. An exact solution is thus possible. This is

$$\chi^x_{mn} v^x_{mn} = -i\omega p^x_{mn} + (i\omega)^2 \sum_r \left\{ \frac{R^{x\ x}_{mrmr}}{B^x_{mr}} p^x_{mn} - \frac{R^{x\ x}_{mnmr}}{B^x_{mr}} p^x_{mr} \right\}$$

where

$$\chi^x_{mn} = B^x_{mn} \left(1 - i\omega \sum_r \frac{R^{x\ x}_{mrmr}}{B^x_{mr}} \right)$$

A similar solution exists for v^y_{mn}. Finally, by substituting these values into equation (5.10) we obtain the approximate result

$$\left(\omega^2 m_p + i\omega \rho_0 c_0 \right) v^F_{mn} = i\omega p^F_{mn} - i\omega^2 \sum_r \frac{T^{F\ x}_{mnmr}}{\chi^x_{mr}} p^x_{mr} - i\omega^2 \sum_q \frac{T^{F\ y}_{mnqn}}{\chi^y_{qn}} p^y_{qn} \quad .$$

Expressions for the radiated power spectral density can now be obtained. We again assume that equation (4.15) applies to the forcing field. The radiated power is the sum of the power radiated by each type of mode separately. We will consider here only the resistive power. Thus

$$\Pi(\omega) = \Pi^x + \Pi^y + \Pi^c + \Pi^F \quad .$$

These terms are estimated separately. We obtain, analogous to equation (4.16) (where now the real part only is implied)

$$\Pi^X = \sum_{m,n}{}' < P^X_{mn} \, v^{*X}_{mn} >$$

$$= \rho_0 c_0 \sum_{m,n}{}' \sigma^X_{mn} \left| \frac{i\omega}{\chi^X_{mn}} \right|^2 \phi^X_{mn}$$

with similar expressions for both $\Pi^Y(\omega)$ and $\Pi^C(\omega)$. The expression for $\Pi^F(\omega)$ involves a double summation:

$$\Pi^F(\omega) = \sum_{m,n}{}' < P^F_{mn} \, v^{*F}_{mn} >$$

$$= \rho_0 c_0 \sum_{m,n}{}' < | v_{mn} |^2 >$$

$$= \rho_0 c_0 \left| \frac{i\omega}{-\omega^2 m_p + i\omega \rho_0 c_0} \right|^2 \times$$

$$\sum_{m,n}{}' \left\{ \phi^F_{mn} + \sum_r{}' \left(T^{F\,X}_{mn\,mr}\right)^2 \left| \frac{\omega}{\chi^X_{mr}} \right|^2 \phi^X_{mr} + \sum_q{}' \left(T^{F\,Y}_{mn\,qn}\right)^2 \left| \frac{\omega}{\chi^Y_{qn}} \right|^2 \phi^Y_{qn} \right\}$$

This expression is simplified by substituting for the coupling coefficients and performing one of the summations in each of the last two terms. The two summations are over the terms k_n^2 and k_m^2, respectively.

If large numbers of acoustically fast modes are excited these summation have the respective values, noting the parity requirement, $\frac{k_o^3 \ell_3}{4\pi}$ and $\frac{k_o^3 \ell_1}{4\pi}$. The radiation excited by the term ϕ_{mn}^{X} taking both edge and acoustically fast modes into account is then found to be

$$\left\{ \frac{\ldots}{A_p \ldots m} + \left| \frac{\ldots \rho_o c_o}{\omega^2 m_p \ldots \rho_o c_o} \right|^2 \frac{H\, k_o \ell_3}{\pi^3 (k_{mn} \ell_3)^2} \right\} \rho_o c_o \left| \frac{\omega}{\chi_{mn}} \right|^2 \phi_{mn}^{X}$$

The direct radiation at resonance is considerably higher than the radiation from the coupled nonresonant acoustically fast modes. We thus obtain as an approximation to the radiated power spectral density the sum of the radiation from each type of mode independe..ly.

$$\Pi(\omega) = \rho_o c_o \sum_{m,n} \left\{ \begin{array}{l} \sigma_{mn}^{X} |Y_{mn}^{X}|^2 \phi_{mn}^{X} + \sigma_{mn}^{Y} |Y_{mn}^{Y}|^2 \phi_{mn}^{Y} \\ + \sigma_{mn}^{c} |Y_{mn}^{c}|^2 \phi_{mn}^{c} + \left| \frac{\ldots}{\omega m_p + i \rho_o c_o} \right|^2 \phi_{mn}^{F} \end{array} \right\} \qquad 5.11$$

The summations are nominally over all modes of each type. However, the reduction of the modal equations to a form we could solve was made on the basis of estimates of the coupling between resonant modes. These modes are picked out when equation (5.11) is averaged over a narrow band of frequencies. The coupling effects, as in the low frequency case, are contained in the admittance functions only. We have gone to

some lengths to show that most of the coupling effects, in particular those between different types of modes, are small. The neglect of the additional coupling terms will be most justified when the structural damping is small. No modal coupling now occurs, except between different modes with the same resonance frequency. We do not include this possibility in the analysis The criteria to be satisfied for there to be no coupling are:

$$\left.\begin{array}{ll} \text{i) no coupling of corner modes if} & \\ & \nu\,\omega\,\eta^{c}_{mn} \ll 1 \\ \text{ii) no coupling of edge modes if} & \\ & \nu^{x}\,\omega\,\eta^{x}_{mn} \ll 1 \\ \text{and} & \nu^{y}\,\omega\,\eta^{y}_{mn} \ll 1 \end{array}\right\} \qquad 5.12$$

where ν^{x} and ν^{y} are the modal densities of those edge modes that are coupled defined by

$$\nu^{x,y}(\omega) = \frac{\ell_{3,1}}{\pi}\,\frac{\partial k_{mn}}{\partial \omega} \approx \frac{\ell_{3,1}}{\pi}\,\frac{k_{mn}}{2\omega} \, .$$

When the above criteria are satisfied, averaging equation (5.11) over a narrow band of frequencies leads to the straightforward result

$$[\text{illegible}] = \frac{\pi}{2M_p}\,\frac{\rho_0 c_0 \sigma^x_{mn}}{\omega M_p \eta^x_{mn} + \rho_0 c_0 \sigma^x_{mn}}\, N^x \phi^x_{mn} + \frac{\pi}{2M_p}\,\frac{\rho_0 c_0 \sigma^y_{mn}}{\omega M_p \eta^y_{mn} + \rho_0 c_0 \sigma^y_{mn}}\, N^y \phi^y_{mn}$$

$$+ \frac{\pi}{2M_p} \sum_{m,n} \phi^c_{mn}\, \frac{\rho_0 c_0 \sigma^c_{mn}}{\omega M_p \eta^c_{mn} + \rho_0 c_0 \sigma^c_{mn}} + \frac{\rho_0 c_0}{(\omega M_p)^2 + (\rho_0 c_0)^2}\, N^F \Delta\omega \langle \phi^F_{mn} \rangle \; . \qquad 5.13$$

In this expression, the edge mode damping and modal correlation functions are treated as uniform for all edge modes, $N^x = N^y = \frac{A_p}{\pi^2} k_o \frac{\partial k_{mn}}{\partial \omega} \Delta\omega$ is the number of resonant edge modes of each type, and $N^F = \frac{A_p k_o}{4\pi}$ is the number of acoustically fast modes. This result is the same as previously obtained light fluid loading results (e.g. references 2 and 4), with the addition of nonresonant acoustically fast modes. The effect of these additional modes is , in general, small as is shown in section 7.

When the criteria (5.12) are satisfied considerable modal coupling occurs. The frequency averages over the admittance functions must now be obtained as in section 4 (equation (4.20), forward). We obtain the result, analogous to equation (4.30) in the low frequency case

$$\ldots = \frac{[\text{illegible}]}{[\text{illegible}]_p \ldots m \ldots + \frac{\pi}{4}(\ldots \eta_{mn}) \frac{m_p}{M_p} \rho_0 c_0 \sigma_{mn}} N \ldots$$

$$+ \frac{\rho_0 c_0 \sigma^{Y}_{mn}}{[\text{illegible}]_p \ldots m_p \eta_{mn} + \frac{\pi}{4}(\ldots \eta^{Y}_{mn}) \frac{m_p}{M_p} \rho_0 c_0 \sigma^{Y}_{mn}} N^{Y} \phi^{Y}_{mn} \qquad 5.14$$

$$+ \frac{\pi}{4 M_p} \sum_{m,n} \phi^{c}_{mn} \frac{\rho_0 c_0 \sigma^{c}_{mn}}{\omega m_p \eta^{c}_{mn} + \frac{\pi}{4}(\omega \ldots \eta^{c}_{mn}) \frac{m_p}{M_p} \rho_0 c \langle \sigma^{c}_{mn} \rangle}$$

$$+ \frac{\rho_0 c_0}{(\omega m_p)^2 + (\rho_0 c_0)^2} N^{F} \Delta\omega \langle \phi^{F}_{qr} \rangle$$

The (m,n) summations and averages here are over all the resonant modes of each type in the $\Delta\omega$ frequency band. The (q,r) average is over all the acoustically fast modes.

The additional modal inertia terms arising from the admittance functions again serve only to modify slightly the modal resonance frequency and hence the determination of which modes are resonant in the $\Delta\omega$ frequency band. Their effect is again small; the important inertia term is the self inertia defined by equation (3.20).

Typical numerical values of equations (5.13) and (5.14) are discussed in section 7.

6. The Intensity Directivity of the Radiated Field

We will discuss briefly the directivity pattern of the intensity of the radiated field. At sufficiently large distances from the plate the far field approximation to the radiated pressure field is

[illegible]

where R is the distance of the field point (x, x_3) from the origin, and I_{1m} is defined by

[illegible]

either sin or cos being used as m is either even or odd, respectively. I_{3m} has a similar form.

The intensity of the acoustic field at the point (x, x_3) is defined as

[illegible]

the only component being along the radius vector from the origin. We will assume that equation (4.15) again applies to the external field and neglect the effect of ϕ^F_{mn}. We then obtain a simple approximate form for the intensity of the acoustic field:

$$I = \frac{\rho_0 c_0 k_0^2}{\pi^2 R^2 A_p} \sum_{m,n} \left| \frac{i\omega}{Z_{mn}} \right|^2 \phi_{mn} \left| I_{1m} \right|^2 \left| I_{3n} \right|^2$$

which, on averaging over a narrow frequency band reduces to

$$I(x_1, x_2, \omega)_{\Delta\omega} = \frac{\rho_0 c_0 k_0^2}{\pi^2 R^2 A_p} \frac{\pi}{2 M_p} \sum_{m,n} \frac{\phi_{mn}}{\omega m_p \eta_{mn} + \rho_0 c_0 \sigma_{mn}} \left| I_{1m} \right|^2 \left| I_{3n} \right|^2 \qquad 6.1$$

The summation in equation (6.1) is over all modes with resonance frequencies in the $\Delta\omega$ frequency band. The expression is valid at all (sub-acoustic critical) frequencies.

The directivity is determined by the terms $|I_{1m}|^2$ and $|I_{3n}|^2$. At low frequencies the radiation is obviously uniform in direction. At sufficiently high frequencies, the dominant radiation is from edge modes. Now, for example, an X-type edge mode ($k_m < k_o$, $k_n > k_o$)

has a sharp maximum intensity of radiation in the direction

$$\frac{x_1}{R} = \pm \frac{k_m}{k_o}$$

We can approximate $|I_{3n}|^2$ here by $2/k_{mn}^2$. For any given mode, the direction of maximum radiation at its resonance frequency is nearer to the normal to the panel under fluid loading conditions because the corresponding k_o is decreased. The change in $\frac{x_1}{R}$ is in the ratio of the change in resonance frequencies, namely

$$\left(\frac{m_p}{M_p}\right)^{1/2} = \left(1 + \frac{\rho_o}{m_p k_{mn}}\right)^{1/2}$$

But, at any given frequency ω, k_o, and hence the "width" of the edge mode regions in $\underline{k}$-space, is fixed. Thus, for example, the X-mode radiation is always due to modes with the same values of k_m, although the values of k_n corresponding to the resonant modes at this frequency will change. The directivity pattern of radiated intensity at high frequencies is thus essentially independent of fluid loading, although the overall magnitude of the intensity will depend on changes in the wavenumbers associated with a resonance frequency, and on the relative numbers of resonant edge modes in a bandwidth.

We note finally that the radiated power is the total intensity integrated over all directions. To demonstrate this we require from equation (6.1) the integral

$$E = \int_{\Omega} |I_{1m}|^2 \, |I_{3n}|^2 \, d\Omega \; ,$$

where Ω is the surface of solid angle 2π at radius R. For the X-edge modes we make the approximation

$$|I_{3n}|^2 = \frac{2}{k_{mn}^2} \; ,$$

and write

$$E^x = \int_{\Omega} |I_{1m}|^2 \, \frac{2}{k_{mn}^2} \, d\Omega$$

$$= \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \; \frac{4 k_m^2 \begin{matrix}\sin^2\\ \cos^2\end{matrix}\left(\frac{k_o \ell_1}{2} \sin\theta\cos\phi\right)}{\left(k_m^2 - k_o^2 \sin^2\theta \cos^2\phi\right)} \; \frac{2}{k_{mn}^2} \; R^2 \sin\theta$$

$$= \frac{32 k_m^2 R^2}{k_{mn}^2} \int_0^{\pi/2} d\theta \int_0^1 \frac{\begin{matrix}\sin^2\\ \cos^2\end{matrix}\left(\frac{k_o \ell_1}{2} x \sin\theta\right)}{\left(x^2 - \frac{k_m^2}{k_o^2 \sin^2\theta}\right)^2} \; \frac{\sin\theta}{k_o^4 \sin^4\theta} \; \frac{dx}{\sqrt{1-x^2}}$$

Using an approximation similar to that used in section 3, namely

$$\frac{\begin{matrix}\sin^2\\ \cos^2\end{matrix}\left(\frac{k_o \ell_1}{2}\, x \sin\theta\right)}{\left(x^2 - \frac{k_m^2}{k_o^2 \sin^2\theta}\right)^2} = \begin{cases} \frac{\pi}{8}\ell_1 \frac{k_o^3 \sin^3\theta}{k_m^2}\, \delta\left(x - \frac{k_m}{k_o \sin\theta}\right), & \sin\theta > \frac{k_m}{k_o} \\ 0 & , \sin\theta < \frac{k_m}{k_o} \end{cases}$$

we can obtain:

$$E^x = + \pi \ell_1 \frac{R^2}{k_{mn}^2} \int_{\sin^{-1}\frac{k_m}{k_o}}^{\pi/2} \frac{\sin\theta \, d\theta}{\left(k_o^2 \sin^2\theta - k_m^2\right)^{1/2}}$$

$$= \frac{2\pi^2 \ell_1 R^2}{k_o k_{mn}^2}$$

with a similar expression for E^y. E^c is easily calculated by approximating $|I_{1m}|^2$ and $|I_{3n}|^2$ as $2/k_m^2$ and $2/k_n^2$, respectively, giving

$$E^c = \frac{8\pi R^2}{k_m^2 k_n^2}$$

Using these values together with equation (6.1) we finally obtain

$$\Pi(\omega, \Delta\omega) = \int_{\Omega} I(x_1, x_2, \omega)\, \Delta\omega \; d\Omega$$

$$= \frac{\pi}{2M_p} \frac{2\rho_0 c_0 k_0 \ell_1}{A_p k_{mn}^2} \sum_{m,n} \frac{\phi_{mn}^X}{\omega m_p \eta_{mn}^X + \rho_0 c_0 \sigma_{mn}^X}$$

$$+ \frac{\pi}{2M_p} \frac{2\rho_0 c_0 k_0 \ell_2}{A_p k_{mn}^2} \sum_{m,n} \frac{\phi_{mn}^Y}{\omega m_p \eta_{mn}^Y + \rho_0 c_0 \sigma_{mn}^Y}$$

$$\frac{\pi}{2M_p} \frac{8\rho_0 c_0}{\pi A_p} \sum_{m,n} \frac{k_0^2}{k_m^2 k_n^2} \frac{\phi_{mn}^C}{\omega m_p \eta_{mn}^C + \rho_0 c_0 \sigma_{mn}^C}$$

This expression is equivalent to equation (5.13) if ϕ_{mn}^F is zero and when suitable expressions are substituted for the modal radiation coefficients; and to equation (4.29) when only corner modes are excited.

7. Evaluation of the Effect of Fluid Loading

In this section some typical values of the power spectral densities of radiated sound obtained in sections 4 and 5 are estimated when the plate is excited by a turbulent boundary layer. Corcos'[7] model is used for the cross-spectral density of the pressure field generated by the turbulent boundary layer. This is

$$\phi(\underline{r},\omega) = \phi(\omega)\exp\left\{-\frac{\omega|r_1|}{\alpha_1 U_c} - \frac{\omega|r_3|}{\alpha_3 U_c} - \frac{i\omega r_1}{U_c}\right\} \qquad 7.1$$

where we have further assumed that the longitudinal and lateral amplitudes decay exponentially. In this expression $\phi(\omega)$ is the spectral density of the mean square boundary layer pressure, $\underline{r} \equiv (r_1, r_3)$ is the spatial separation, U_c is the convection speed of the pressure field and α_1 and α_3 are non-dimensional constants. In reference 4 it is shown that the modes are uncoupled (i.e. condition 4.15 holds) if

$$\frac{\alpha_1 U_c}{\omega} \ll \ell_1 \quad , \quad \frac{\alpha_3 U_c}{\omega} \ll \ell_3$$

It is assumed that these inequalities are satisfied for the range of frequencies we will consider. It is further assumed that U_c is so small that no hydrodynamic coincidence effects occur. This last assumption, although reasonable, is made solely to make the determination of ϕ_{mn} easy and does not affect the discussion of the fluid loading effects. Under these assumptions we obtain the simple relation:

$$\phi_{mn}(\omega) = \phi(\omega) \frac{4\alpha_1\alpha_3}{1+\alpha_1^2} \frac{U_c^2}{\omega^2} . \qquad 7.2$$

ϕ_{mn} is thus constant for all modes being a function only of frequency.

The structural damping of the plate is an important factor in determining the velocity response and the associated radiation. Unfortunately no simple description, if any, of the structural damping is available. Most of the energy dissipated in thin plates is into the end supports and thus depends greatly on the means of support and the structures to which the plate is attached. The effect of the structural damping is further modified under dense fluid loading conditions by its magnitude relative to the radiation damping. The addition of damping treatment to a plate can lead to different effects under light and dense fluid loading conditions. To demonstrate the interaction of both structural and fluid damping, and the magnitudes of the fluid loading effects, some extremely simplyfying assumptions will be made about the structural damping. We will assume that the structural loss factor is inversely proportional to frequency and is independent of wavenumber. This last assumption is equivalent to assigning the total structural loss factor measured in narrow frequency bands to each mode in the band. For thin plates the total structural loss factor is found in general to be inversely proportional to frequency, but is usually measured only

over fairly small ranges of frequency. We will partly compensate for this lack of knowledge of values of the loss factor by evaluating the power for several widely different values. We thus write for all modes

$$\eta_{mn} = \eta = \frac{\beta}{\omega}$$

where β is constant, and evaluate the radiated power using various values of β.

Under these simplyfying assumptions, the evaluation of the radiated power spectral density is particularly straightforward. For example, at low frequencies, by averaging over all modes in a band, we obtain from equation (4.29)

$$\Pi(\omega) = \frac{\pi}{2 m_p} \phi(\omega) \frac{4 \alpha_1 \alpha_3 \; U_c^2}{(1+\alpha_1^2)\, \omega^2} \left\{ \frac{\nu}{\left(1 + \frac{\rho_0}{m_p \kappa_{mn}}\right)} \left(1 + \frac{\pi \nu \beta}{\rho_0 c_0 \langle \sigma_{mn} \rangle} \cdot \frac{k_{mn}(\ell_1 + \ell_3)}{\pi^2}\right)^{-1/2} \right\}$$

and from equation (4.30):

$$\Pi(\omega) = \frac{\pi}{2 m_p} \phi(\omega) \frac{4 \alpha_1 \alpha_3 \; U_c^2}{(1+\alpha_1^2)\, \omega^2} \left\{ \frac{\nu}{\left(1 + \frac{\rho_0}{m_p k_{mn}}\right)} \left(\frac{m_p \beta}{\rho_0 c_0 \langle \sigma_{mn} \rangle} + \frac{\pi}{4} \frac{\nu \beta}{\left(1 + \frac{\rho_0}{m_p k_{mn}}\right)} \right)^{-1} \right\}$$

The average values of the radiation coefficients obtained in reference 4 are used. These differ by a factor 2 from those published earlier

by Maidanik[2]. No account will be taken of the second order inertia terms when alculating the modal resonance frequences. In the calculations performed here, the wavenumber corresponding to resonant modes at any frequency (including the modal self-inertia term), that is, the solution of the equation

$$D k_{mn}^{5} - \omega^{2} m_{p} k_{mn} - \omega^{2} \rho_{0} = 0$$

was obtained graphically.

Typical values of these expressions are compared in Figure 3 with the spectral density obtained by neglecting fluid loading, that is, with the expression

$$\Pi(\omega) = \frac{\pi}{2 m_{p}} \phi(\omega) \frac{4 \alpha_{1} \alpha_{3} U_{c}^{2}}{(1+\alpha_{1}^{2}) \omega^{2}} \left\{ \nu \frac{\rho_{0} c_{0}}{m_{p} \beta} \langle \sigma_{mn} \rangle \right\}$$

where here the modal expressions are evaluated at the in vacuo resonant frequencies. The expressions obtained in section 5 can be simplified in a similar manner.

Figures 5 and 6 show some typical values of the power radiated by a 2' x 2' x 1/10" steel plate water loaded on one side. A very thin plate is chosen to demonstrate more markedly the effects of fluid loading. For thicker plates the modal density is too low for the averaging technique over resonant modes in narrow frequency bands to be

applicable at low frequencies (assuming 1/3 octave bands are used), although the radiation may be readily computed from equation (4.18). Three different values of β are used, namely, $\beta = \pi$, 20π and 200π. These values correspond to quality factors at 1000 Hertz of 2000, 100 and 10, respectively. The range thus covers both very lightly and very heavily damped systems. We would expect the value $\phi = 100$ at 1000 Hertz to be of most practical interest.

The curves shown in Figures 5 and 6 are not restricted to the low and high frequency regimes considered in sections 4 and 5 (in this example, these regimes correspond to $f < 625$ Hz and to $f >> 1250$ Hz). Rather it is assumed that the corner modes are always essentially uncoupled to other types of modes and that the corner mode solution is thus applicable at all frequencies. To this solution must be added the edge mode radiation when $k_o\ell_1, k_o\ell_3 > 3\pi$. (In reference 4 it is noted that the first few edge modes excited as one considers increasing frequencies have radiation coefficients more characteristic of corner modes. This fact has been used to estimate the un-loaded radiation and accounts for the discontinuities in the spectra at $k_o\ell_1, k_o\ell_3 = 3\pi$. The edge mode radiation under dense fluid loading is shown dashed in Figure 5 for frequencies $\pi < k_o\ell_{1,3} < 3\pi$.) Except for the very heavily damped case, the edge mode radiation is less than the corner mode radiation until many edge modes are excited. This fact, together with the fact that the contribution from the acoustically fast modes due to ϕ^F_{mn} is negligible (in this example the a.f. radiation

is less than 5% of the total radiation at all the frequencies considered) lends added justification for the neglect of additional coupling effects in the corner mode solution in the middle range of frequencies.

The effect of an increase in the structural damping on an un-loaded plate is a uniform reduction in the radiated spectrum. The effect on a fluid-loaded plate is less straightforward. We consider first the corner mode radiation in Figure 6. For $\beta = \pi$, a very lightly damped plate, the additional damping due to fluid loading causes a marked decrease in the radiated spectrum. For greater damping, $\beta = 20\pi$, the additional effect of fluid loading is not as marked. Both these cases have been evaluated using equation (4.29) (and its analogy at higher frequencies). However, further increase of the structural damping, $\beta = 200\pi$, causes modal coupling as the resonance peaks overlap. Equation (4.30) is now applicable. Figure 6 shows that rather than a further decrease in the fluid loading effect as β is increased from 20π to 200π, the modal coupling results in a greatly increased radiation damping and hence shows a marked fluid loading effect. Further increase of β, of course, would again result in a decreasing change due to fluid loading. This behavior is characteristic of corner mode radiation at all frequencies.

The radiation damping of edge modes is very high. For a 2' x 2' steel plate in water the uncoupled radiation damping of an edge mode corresponds to a quality factor of 20 at 2000 Hz. (This value is independent on the thickness of the plate only through the change of resonance

frequency due to mass loading.) Unless the structural damping is very high, the factor $\rho_o c_o \sigma_{mn}(\omega m_p \eta_{mn} + \rho_o c_o \sigma_{mn})^{-1}$ is essentially unity for all edge modes. The increased radiating efficiency over the corner modes is thus offset by the increased damping and, as seen in Figure 5, very many edge modes must be excited before the edge mode radiation dominates the corner mode radiation. This is true for $\beta = \pi$ and $\beta = 20\pi$. At very high levels of damping however, $\beta = 200\pi$, the increased corner mode radiation damping due to modal coupling decreases the corner mode radiation to such an extent that edge mode radiation again always predominates. The large discontinuity in the curves for $\beta = 200\pi$ is presumably due to our inability to analyise completely the modal coupling effects in the middle range of frequencies. As has been previously remarked the analysis given in section 4 and 5 is most accurate for low values of the structural damping.

[illegible]. Conclusion

The modal approach used here is perhaps not particularly suited to a study of the effects of fluid loading as it leads to an infinite set of linear equations which can, at best, be solved only very approximately. However, the approach has the advantage that being based on the in vacuo modal admittances it allows a direct comparison of the fluid loaded and un-loaded responses of the system. In particular the roles played by both the structural and the radiation damping can be distinguished.

At low frequencies, $k_o\ell_1$, $k_o\ell_3 < \pi/2$, the approximations to the coupling coefficients of the modal equations are good and the resulting equations can be solved fairly accurately. The radiated power spectral density can be obtained either by averaging over resonant modes in narrow frequency bands when the modal density is high (equation (4.20) and (4.30)) or computed directly from equation (4.18).

At higher frequencies, the complexities of the modal interactions are such that considerable simplification of the modal equations must be made before even an approximate solution is found. The approximations made in this case are more justifiable in cases where the structural damping is so low that in any case a negligible amount of modal interaction occurs.

The main effects of fluid loading have been discussed in section 7. We have noted that the inertia coupling terms play a somewhat minor role. This is not really surprising: there can be no exchange of energy via inertial coupling. We have treated systems such that the response of the system at any frequency is described by the response of those modes that are resonant at, or near, that frequency. The response is thus primarily determined by the amplitudes of the modal resonances. These amplitudes are changed from the in vacuo case solely by the additional energy loss by acoustic radiation to infinity: we call this additional energy loss the radiation damping of the plate. The slight changes in resonance frequencies caused by the inertia coupling are overshadowed by the large changes in frequency caused by the modal self inertia.

The magnitude of the radiation damping is partly determined by the amount of modal interaction that occurs. When the structural damping is small the resonance peaks are very sharp; no modal interactions occur as the energy of the system is contained in very narrow, separate bands of frequencies. Thus, the acoustic field at any frequency is generated solely by the mode that is resonant at that frequency. The radiation damping of each mode can then only be due to the acoustic field generated by that mode alone. Except for very thin plates, this is the most typical situation met with in practice. When the structural damping is large, the widths of the resonance peaks are increased, i.e., each mode is considerably excited over a wider band of frequencies. The acoustic field at any frequency is now due not only to the mode resonant at that frequency but also to other modes with resonance frequencies near to that frequency. The radiation damping is correspondingly higher. The net result, of course, is a decrease in the total radiated field because of the decreased amplitude of the plate response.

The two important features of fluid loading are the fluid inertia effect and the radiation damping. Both effects lead to a decrease in the radiated acoustic field. The radiation damping causes a decrease in the velocity response amplitude of the plate. The fluid inertia causes a decrease in the modal resonance frequencies. At any frequency the resonant modes correspond to higher wavenumbers, and thus, have lower radiation efficiencies.

APPENDIX

It has been assumed throughout the analysis that condition 4.15 applies to the external force, that is, there is no modal coupling induced by the applied force. However, at low frequencies a simple expression for the radiated power similar to equation (4.18) can still be obtained even if this condition does not hold. The many cross-coupling terms representing power flow between the modes again cancel and we obtain the expression

$$H(\omega) = \sum_{m,n} \sum_{q,r} R_{mnqr} \, Y_{mn} Y_{qr}^{*} \langle p_{mn} p_{qr}^{*} \rangle \,, \qquad \text{A.1}$$

where we have made use of the approximate result 4.14.

The spectrum of the radiated power is represented by the real part of equation (A.1). This expression can be averaged over a narrow band of frequencies in cases where the modal density is high. We require an estimate of the integral

$$\int_{\omega-\frac{\Delta\omega}{2}}^{\omega+\frac{\Delta\omega}{2}} Y_{mn}(\omega) \, Y_{qr}^{*}(\omega) \, d\omega$$

Using the notation of section 4, and assuming light structural

damping, this integral has the form

$$M_p^2 \int_{-\varepsilon_1}^{\varepsilon_1} \left(\varepsilon + \frac{i m_p}{M_p} \frac{\eta_{mn}}{2} + \frac{i \rho_0 c_0}{\omega M_p} \frac{\sigma_{mn}}{2} \right)^{-1} \left(\varepsilon + \lambda_{qr}^{mn} - \frac{i m_p}{M_p} \frac{\eta_{qr}}{2} - \frac{i \rho_0 c_0}{\omega M_p} \frac{\sigma_{qr}}{2} \right)^{-1} d\varepsilon$$

$$\approx \quad \frac{1}{2 \omega M_p^2 \varepsilon_1}$$

This result is a factor

$$\frac{\omega m_p \eta_{mn} + \rho_0 c_0 \sigma_{mn}}{\omega M_p \varepsilon_1}$$

times the corresponding integral over $|Y_{mn}|^2$. We assume that η_{mn}/ε_1 is always very small, that is, the widths of the modal resonance peaks are less than the width of the frequency band over which we integrate. The corner mode results given in section 4 will thus be approximately correct for low values of the structural damping even when condition 4.15 does not hold. For excitation by a turbulent boundary layer the conditions for equation (4.15) to hold are frequency dependent. It is at low frequencies that the additional approximation given here may be applicable.

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Figure 1

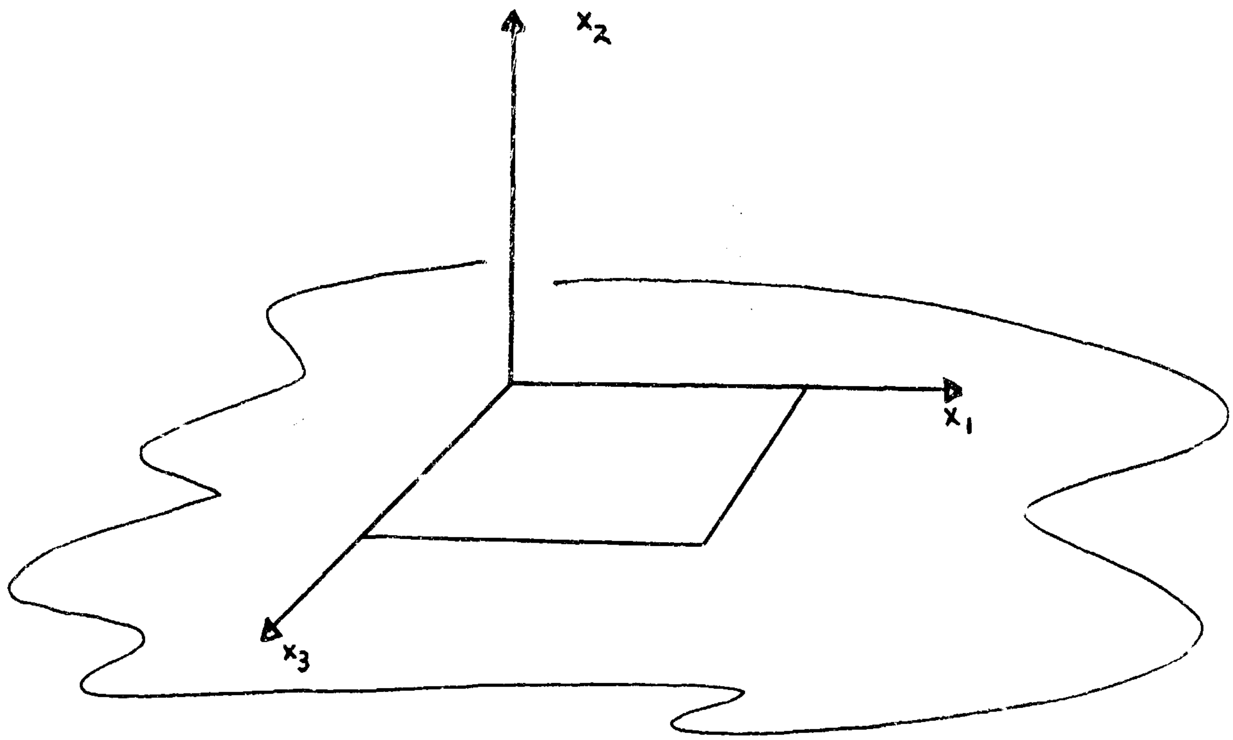
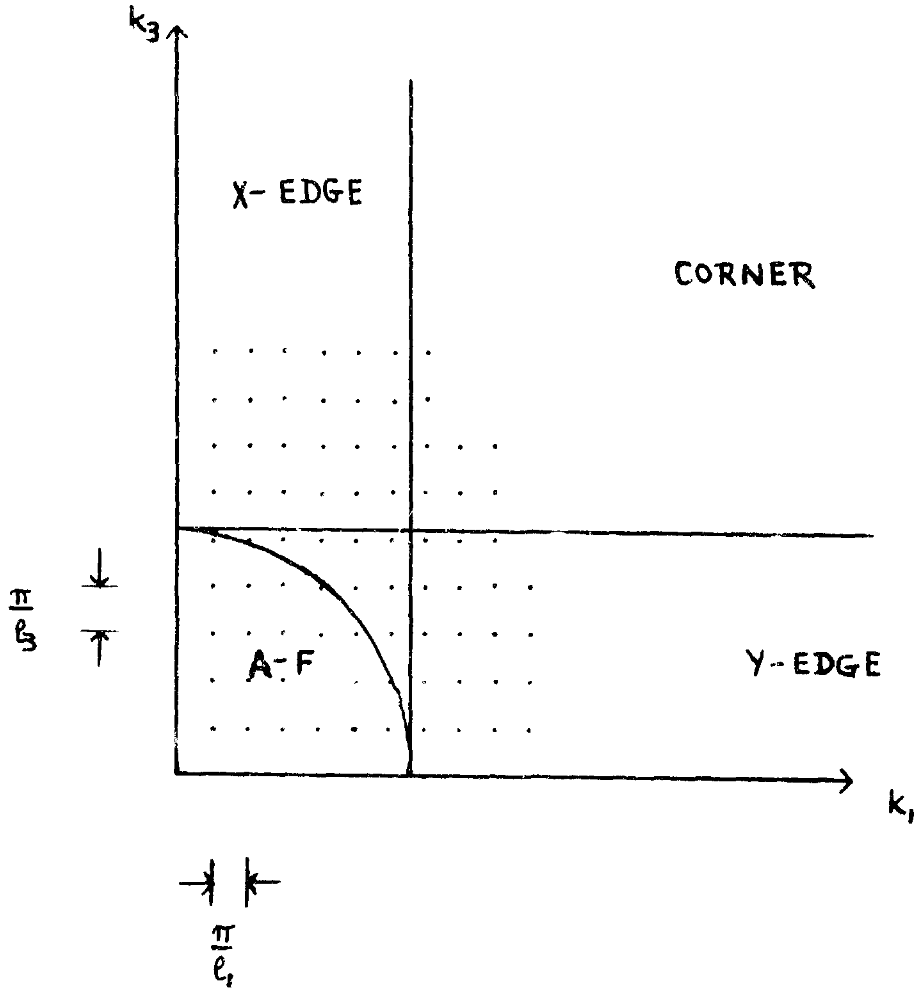


PLATE WITH COORDINATE SYSTEM

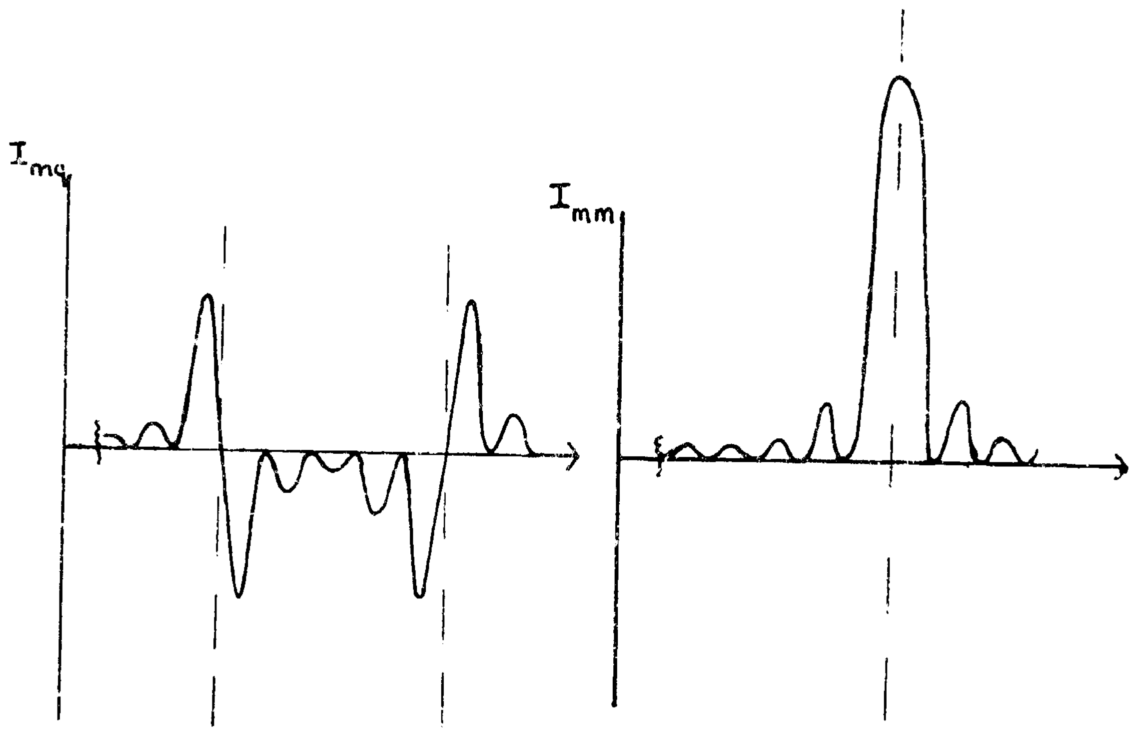
THE DENSE FLUID OCCUPIES THE SPACE $x_2 > 0$.

FIGURE 2



MAIDANIK'S CLASSIFICATION OF MODES IN WAVENUMBER SPACE ACCORDING TO RADIATION CHARACTERISTICS

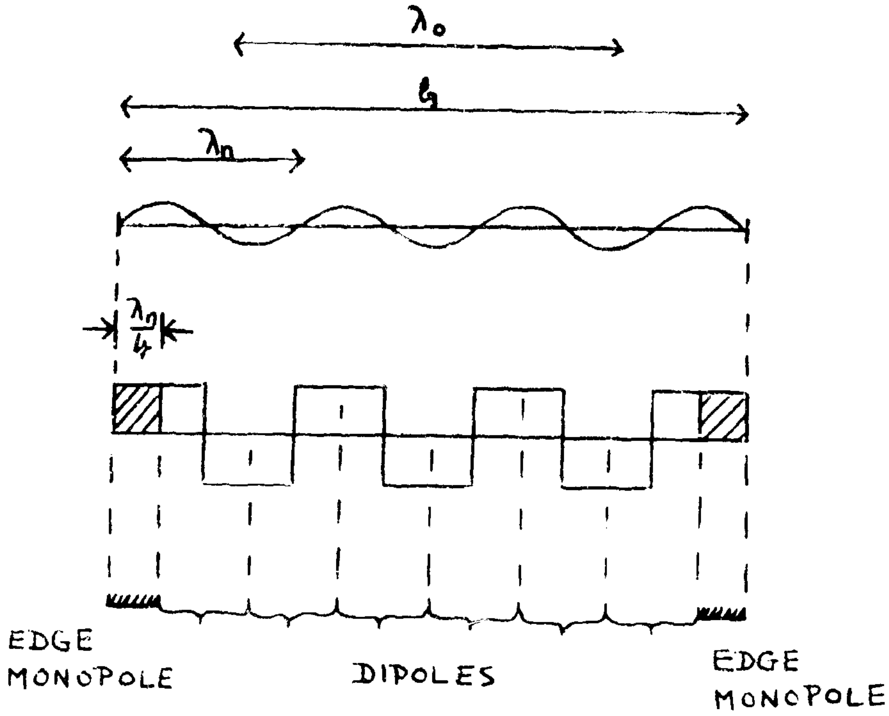
Figure 3



$k_1 = k_m$ $k_1 = k_q$ $k_1 = k_m$

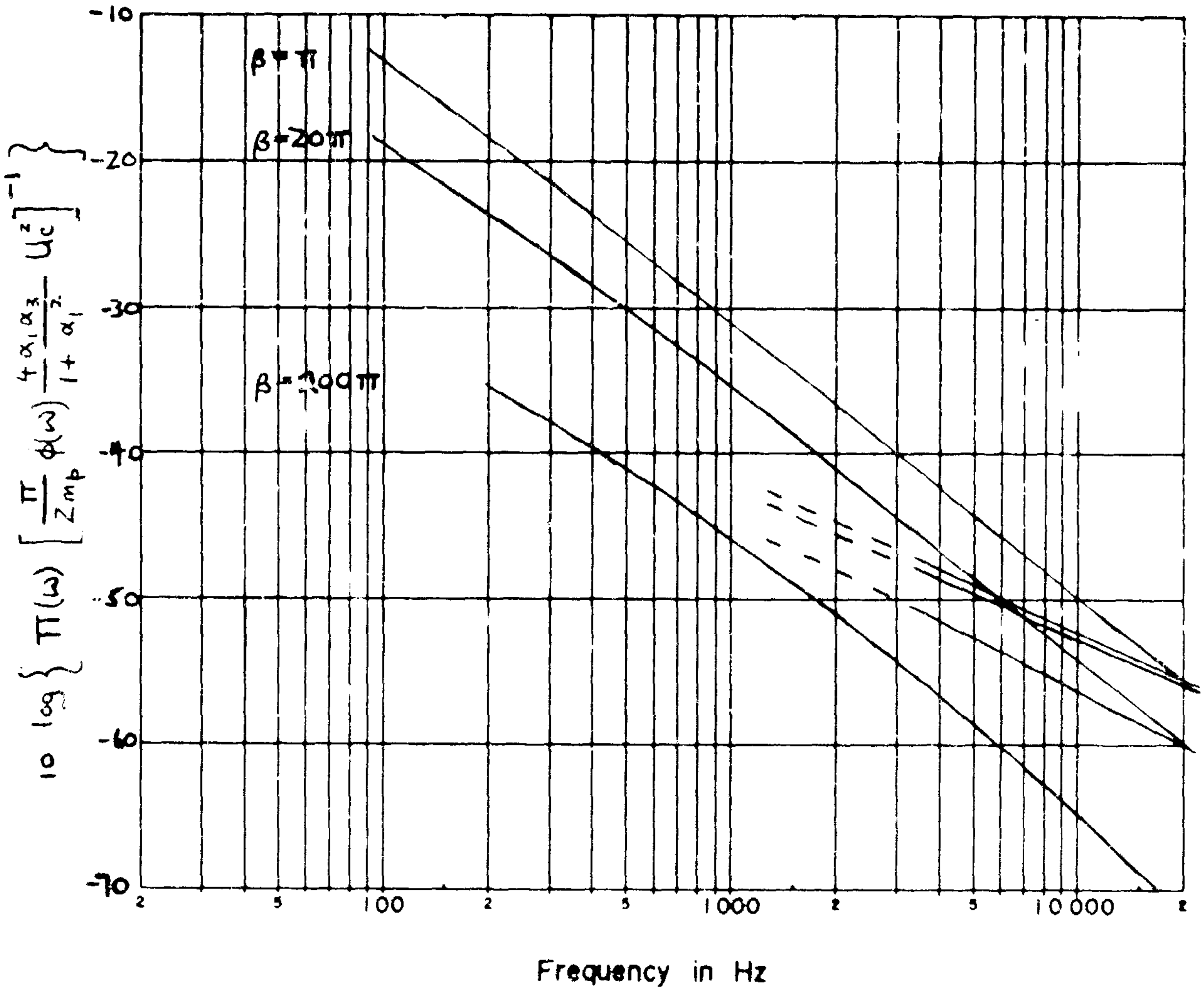
$$\text{GRAPH OF } I_{mq}(k_1) = \frac{(1 - (-1))^m \cos k_1 \ell_1)}{(k_1^2 - k_m^2)\,(k_1^2 - k_q^2)}$$

FIGURE 4



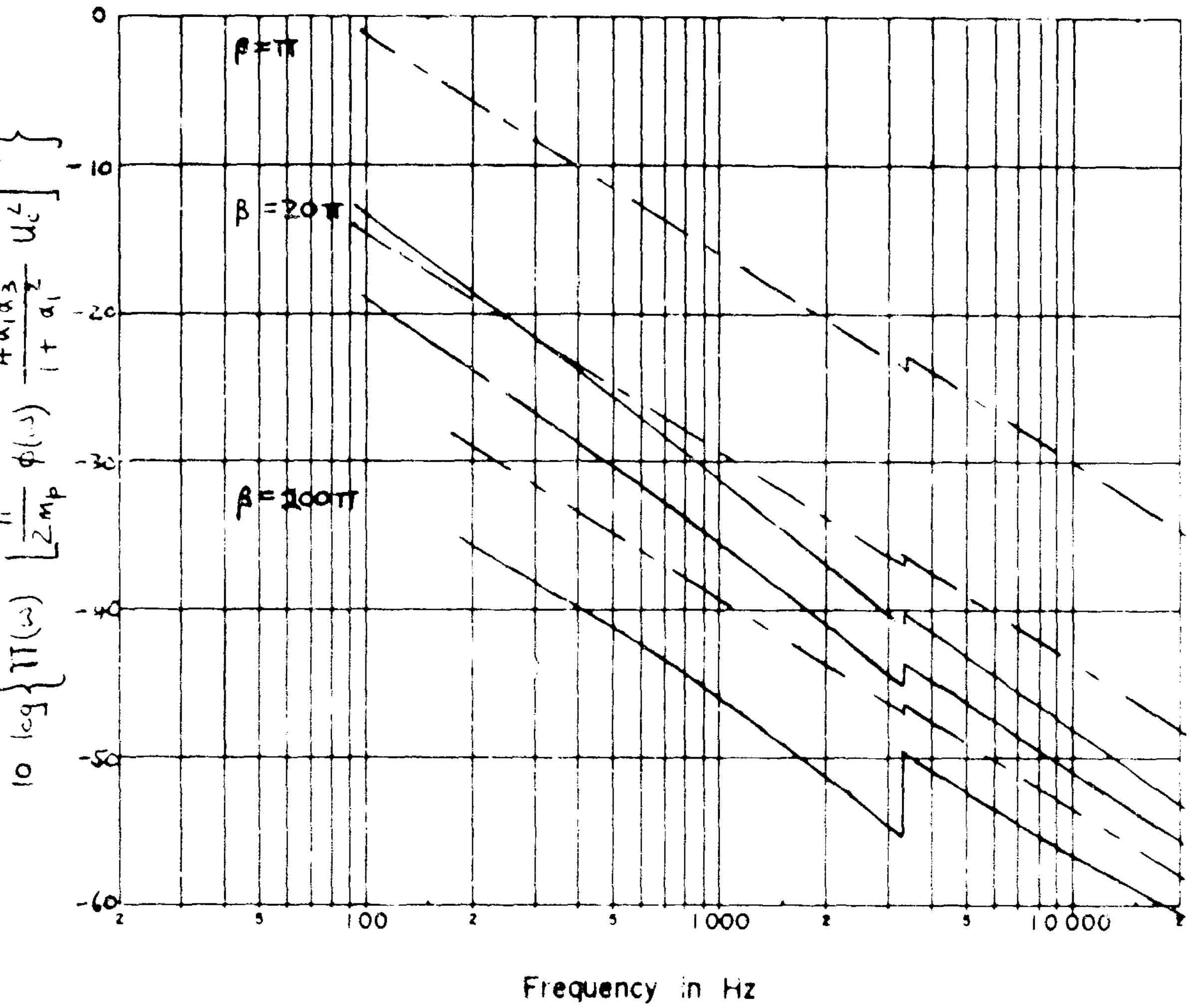
MODAL VOLUME VELOCITY CANCELLATION FOR $\lambda_n < \lambda_o < \ell_3$ (FOLLOWING MAIDANIK[2]), SHOWING THAT RADIATION FROM CENTRE REGION IS AT BEST DIPOLE

FIGURE 5



RADIATED POWER SPECTRAL DENSITY SHOWING SEPARATE CORNER AND EDGE MODE CONTRIBUTIONS FOR VARIOUS β

FIGURE 6



RADIATED POWER SPECTRAL DENSITY: COMPARISON WITH THAT OBTAINED NEGLECTING FLUID LOADING

—— – —— NO LOADING

———— WATER LOADING

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13. ABSTRACT

The acoustic radiation into a fluid filled infinite half-space from a randomly excited, thin rectangular plate inserted in an infinite baffle is discussed. The analysis is based on the in vacuo modes of the plate. The modal coupling coefficients are evaluated approximately at both low and high (but below acoustic critical) frequencies. An approximate solution of the resulting infinite set of linear simultaneous equations for the plate modal velocity amplitudes is obtained in terms of modal admittances of the plate-fluid system. These admittances describe the important modal coupling due to both fluid inertia and radiation damping effects. The effective amount of coupling, and hence the effective radiation damping acting on a mode, depends on the relative magnitudes of the structural damping, i.e., on the widths of the modal resonance peaks, and the frequency spacing of the resonances. Expressions are obtained for the spectral density of the radiated acoustic power for the particular case of excitation by a turbulent boundary layer.

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